THE NEW ELEMENTARY

CO-ORDINATE GEOMETRY

(For—I Year Degree Students of J. & K University)

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CONTENTS

Chapter P

I. **Co-ordinate axes** ...
Co-ordinates of a point ...
Distance between two points ...
Section formulae ...
Area of the triangle ...

II. **Locus** ...

III. **The Straight line** ...
Slope of a line ...
Intercepts on the axes ...
Other Tests for the Collinearity of three points ...

IV. **Angle between two st. lines** ...
Condition of parallelism ...
Condition of perpendicularity ...
Determination of a st. line ...

V. **Intersection of st. lines** ...
Concurrence of three lines ...
Special test for concurrence of three lines ...

VI. ✓**Length of the Perpendicular** ...
✓Bisectors of angles between two straight lines ...

VII. ✓**Joint Equation of two or more st. lines** ...

VIII. ✓**The Circle** ...

Chapter ✓ *Pa*

IX. **Tangents and Normals** ...
Equation of the tangent ...
Equation of the normal ...

X. **A Circle and a Point** ...
Chord of contact ...
Pole and Polar ...
Properties of Pole & Polar ...

XI. **Two or More Circles** ...
Orthogonal circles ✓ ...
Common tangents to two circles ...
Radical axis ✓ ...
Co-axal circles ✓ ...
University Papers ...
Answers

CHAPTER I

CO-ORDINATE AXES

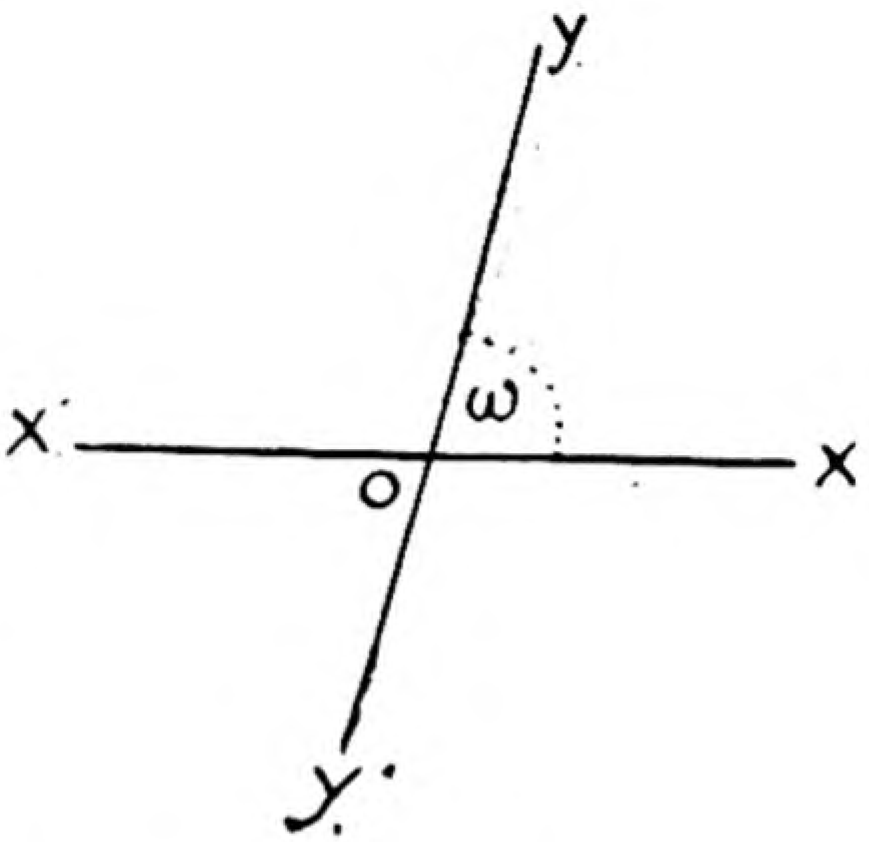
1.1. Let XOX′ and YOY′ be two fixed straight lines inclined with each other at an angle ω (read as 'omega'). Let X′OX and Y′OY be the positive directions of these two lines. Then

(*i*) X′OX is called the *Axis of* x (or, shortly, the *x-axis*),

(*ii*) Y′OY is called the *Axis of* y (or, shortly, the *y-axis*),

(*iii*) X′OX and Y′OY *taken together* are called the *Co-ordinate Axes,*

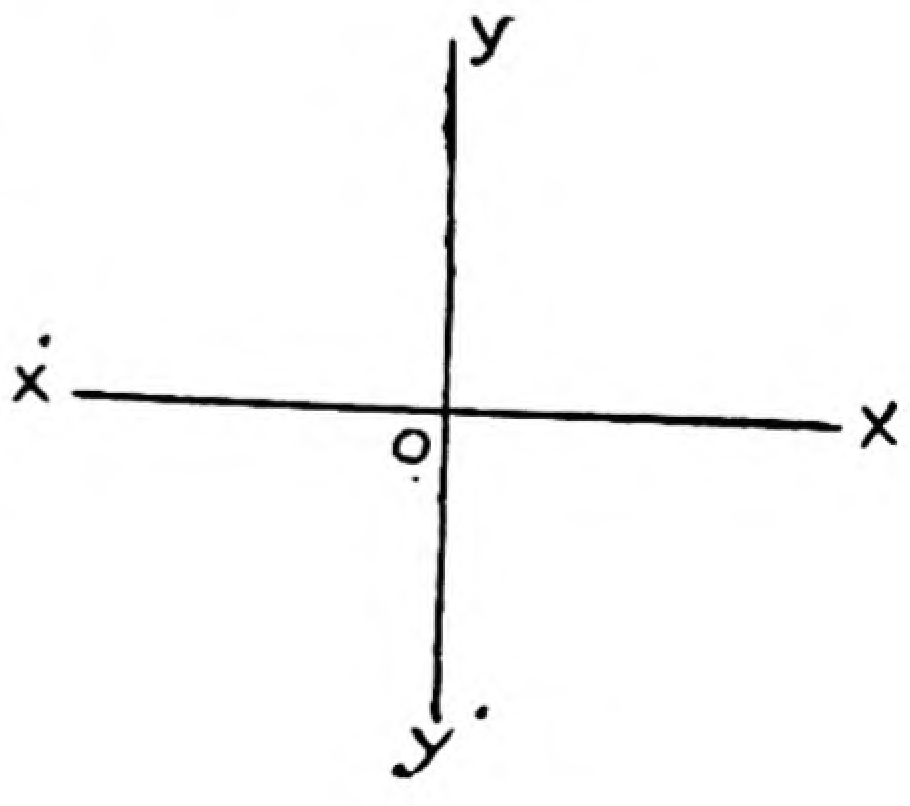
(*iv*) O is called the *origin,* and



(*v*) ω is called the *angle of inclination* of the axes.

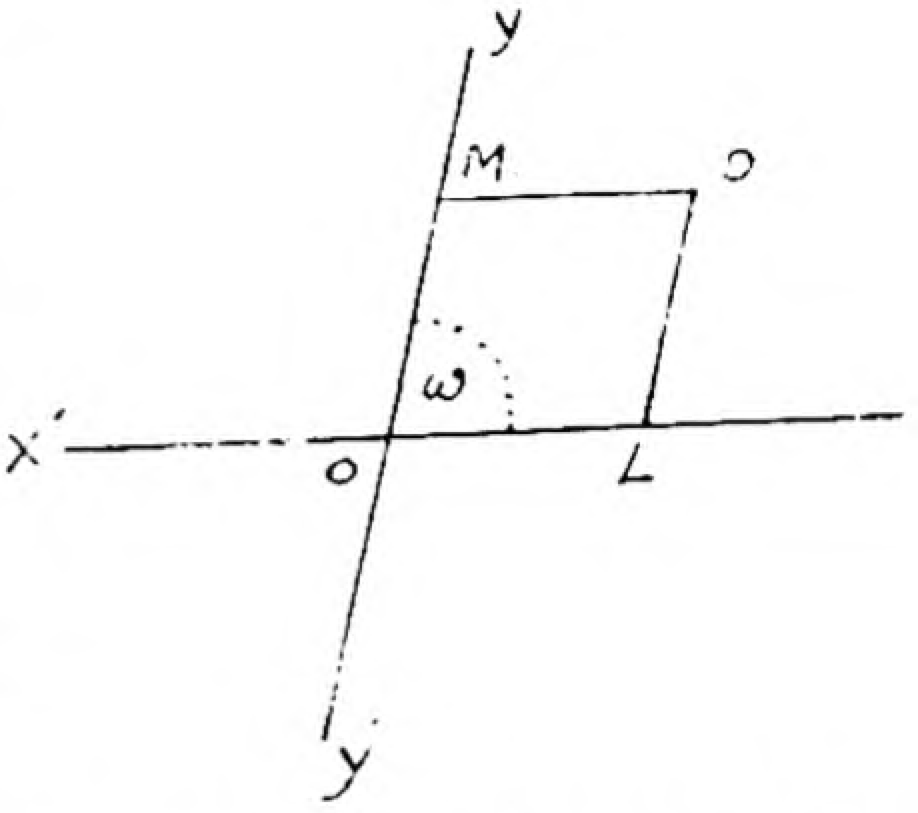
1.11. Rectangular Co-ordinate Axes.

If the axes X′OX and Y′OY are inclined at *right angles* with each other, *i.e.*, if $\omega=90°$, then these are called the *Rectangular Co-ordinate Axes*; otherwise, these are called the *Oblique Co-ordinate Axes.*



1.2. Co-ordinates of ω point.

Let P be any point in the plane. Draw PL and PM parallel to OY and OX respectively meeting OX in L and OY in M. Then the distance OL is called the *Abscissa* (or, the *x—co-ordinate*) and the distance OM, the *ordinate* (or, *the y—co-ordinate*) of the point P respectively. Both the distances OL and OM *taken together* are called the *co-ordinates* of the point P.



If OL$=x'$ and OM$=y'$, then x' and y', are the co-ordinat
of P, and these are, for the sake of brevity, denoted by tl
symbol (x', y').

Note : The abscissa x' is, as a matter of convention, writt
first, and the ordinate y', *after* x', the two being separated by
comma.

1.3. Co-ordinates of a point when the axes a
rectangular.

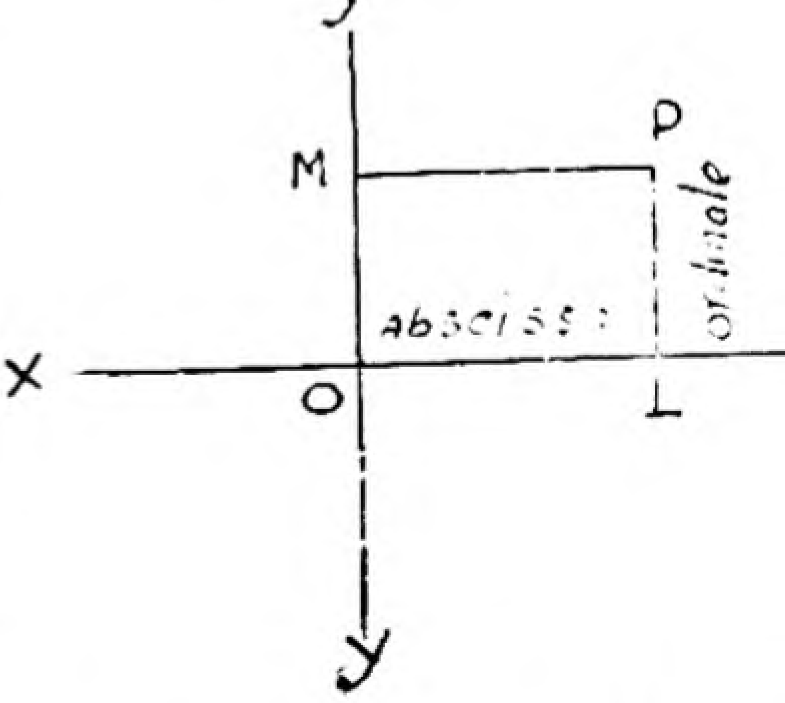
Let P be any point in the plane of the rectangular axes.

From P draw PL ⊥ on OX and PM ⊥ on OY. Then

(*i*) OL is called the *abscissa* (or, *x—co-ordinate*) of P,

(*ii*) PL is called the *ordinate* (or, *y—co-ordinate*) of P,

(*iii*) Both the *abscissa* and *ordinate* of P taken together are called the *co-ordinates* of P.



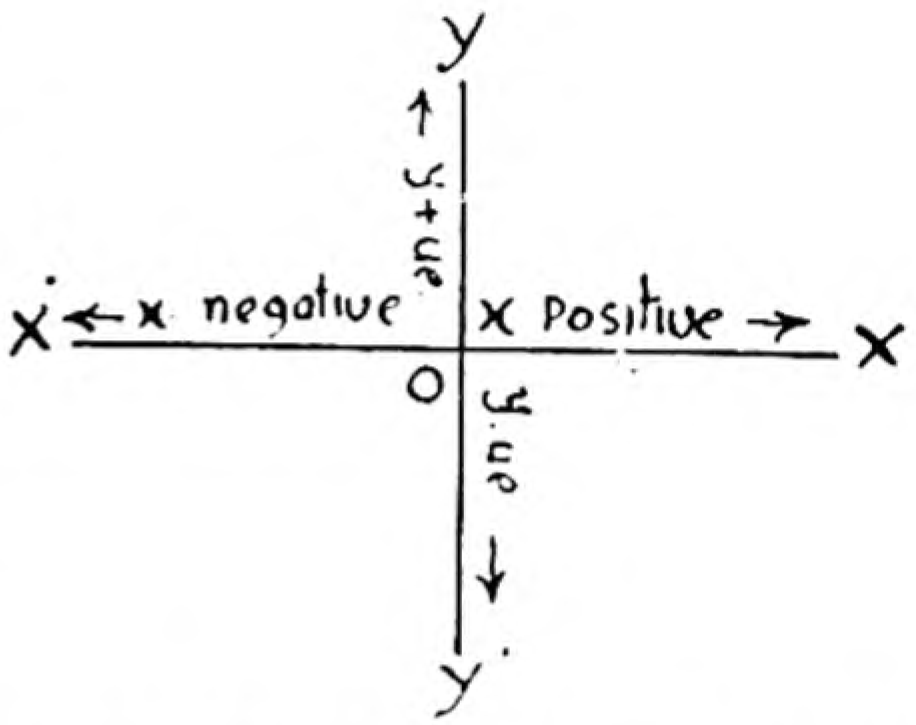
If OL$=x$ and PL$=y$, then x and y are the *co-ordinate*
P, and, for the sake of brevity, these are denoted by the sym
(x, y).

Note : *Throughout this book, we shall deal with 'rectangular axes' and with 'rectangular co-ordinates' as such. Articles on the "Oblique Axes" also will, however, be explained wherever necessary.*

1.4. Signs of Co-ordinates.

(*i*) x, or the abscissa, is *positive* if measured to the *right* of the *origin*, and *negative*, if measured to its *left*.

(*ii*) y, or the ordinate, is *positive* if measured *above* the *x-axis*, and *negative* if measured *below* it.



Note : (*i*) The point whose co-ordinates are (x, y) is *shortly* written as $P(x, y)$. As a matter of convention, x is written first within the brackets and y after x, the two being separated by a comma.

(*ii*) P(0, 0) denotes the *origin* itself.

1.5. Quadrants.

The two mutually perpendicular lines XOX′ and YOY′ divide the entire plane into four parts called *Quadrants*.

(*i*) XOY is called the *first quadrant*,
(*ii*) X′OY the *second quadrant*,
(*iii*) X′OY the *third quadrant*, and
(*iv*) XOY′ the *fourth quadrant*.

1.51. Signs of co-ordinates in different quadrants

(*i*) *First Quadrant.*

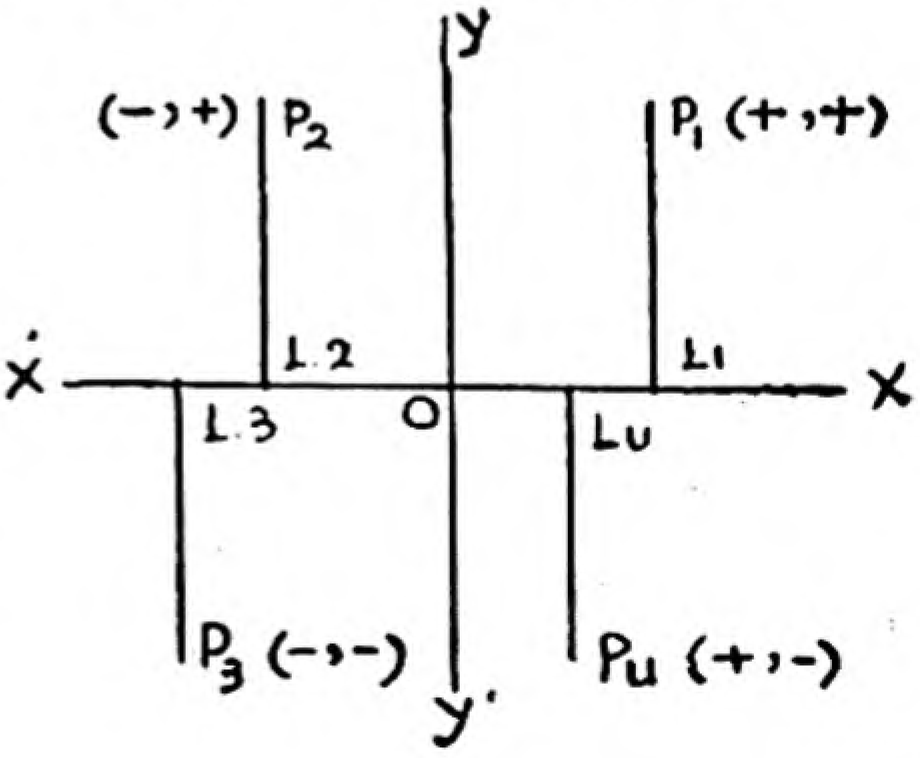
The co-ordinates of P_1 are $(+, +)$, as both OL_1 and P_1L_1 are *positive*.

(*ii*) *Second Quadrant.*

The co-ordinates of P_2 are $(-, +)$, as OL_2 is $-$ve and P_2L_2 is $+$ve.

(*iii*) *Third Quadrant.*

The co-ordinates of P_3 are $(-, -)$, as OL_3 is $-$ve and P_3L_3 is also $-$ve.



(*iv*) *Fourth Quadrant.*

The co-ordinates of P_4 are (+, −), as OL_4 is +ve a P_4L_4 is −ve.

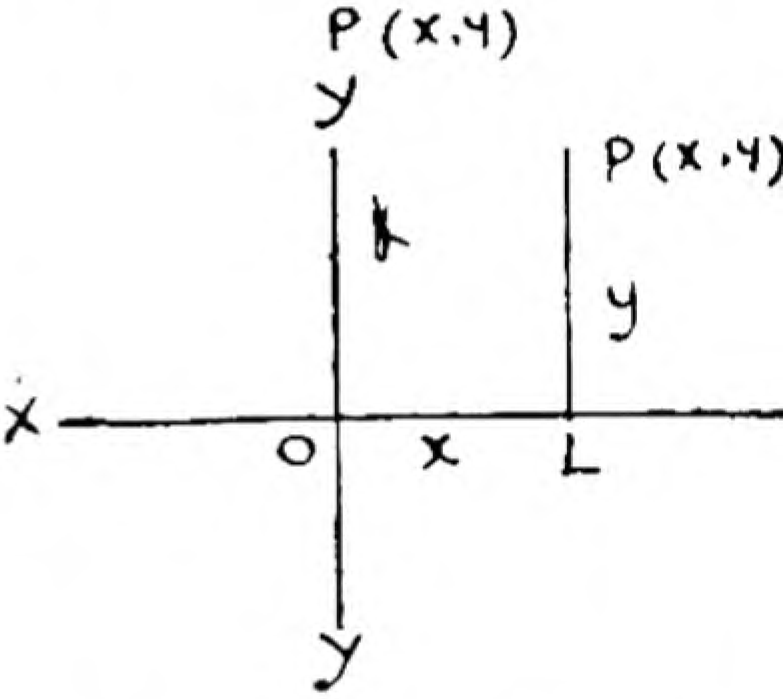
Example. The points (4, 5), (−4, 2). (−5, −7) [a (3, −4) will be in the first, second, third, and fourth quadra respectively.

1.6. How to plot a point P(x, y).

(*i*) *Measure OL along the x-axis and =x, observing, of course, the rule for sign of x, as given in Article 1.4.*

(*ii*) *Measure PL ⊥ to the x-axis and =y, the rule for tke sign of y being observed in terms of Article 1.4.*

Then P is the required point (x, y).



Exercise. Plot the points (3, 5), (−5, −1), (0, 3), (−4, (0, 0). (−3, 0).

1.7. Distance between two points.

To find the distance between two points whose co-ordinates are given.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points. Join PQ and let PQ = *d*.

Draw PL and QM perpendiculars on OX and PR perpendicular on QM.

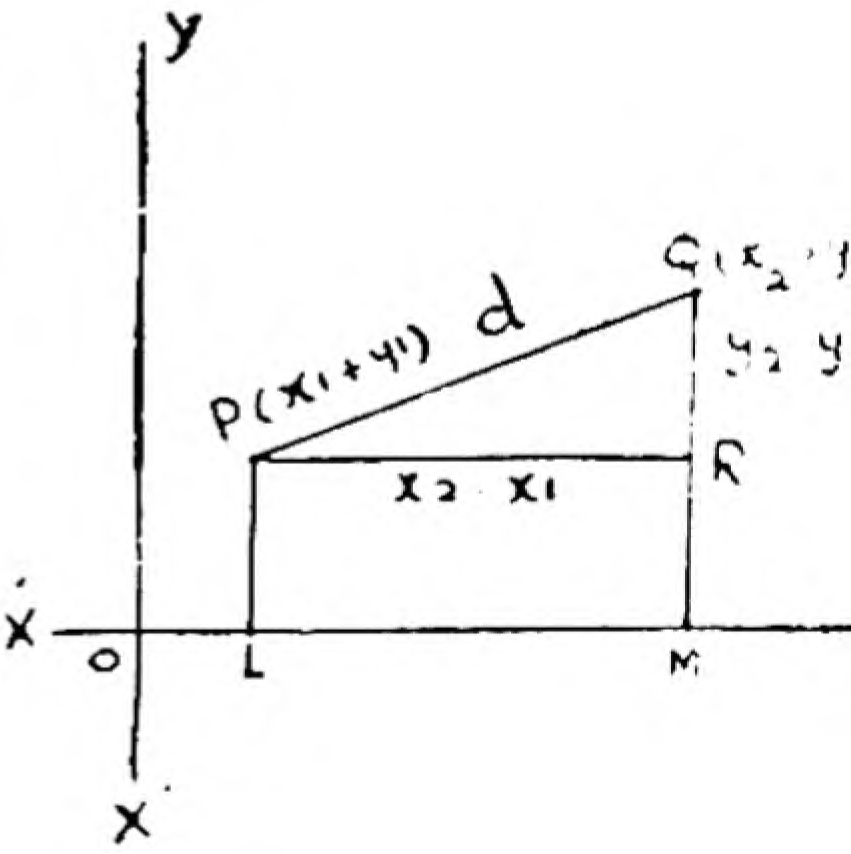
Now $PR = LM = OM - OL$

$= x_2 \quad x_1$ ...(*i*)

$QR = QM - RM$

$= QM - PL$

$= y_2 - y_1$ ...(*ii*)



∴ From the right-∠d △ PQR, we have

$$PQ^2=PR^2+QR^2$$

or $$d^2=(x_2-x_1)^2+(y_2-y_1)^2 \quad \text{[from (i) and (ii)]}$$

∴ $$\mathbf{d}=\sqrt{(\mathbf{x}_2-\mathbf{x}_1)^2+(\mathbf{y}_2-\mathbf{y}_1)^2}.$$

Note : (*i*) Actually, $d=\pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, but as distance between two points means distance in *magnitude*, we reject the —ve sign and keep the +ve sign only.

(*ii*) The distance formula may also be taken as $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, as $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ *is the same* as $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

(*iii*) In the above article, P(x_1, y_1) and Q(x_2, y_2) have been taken in the *first quadrant*, but the proof of this article or that of any subsequent article is true in whatever quadrant the points may lie. In fact, henceforth we shall presume all the points to lie in the *first quadrant.*

Cor. To find the distance of the point P($\mathbf{x}_1$, $\mathbf{y}_1$) from the origin.

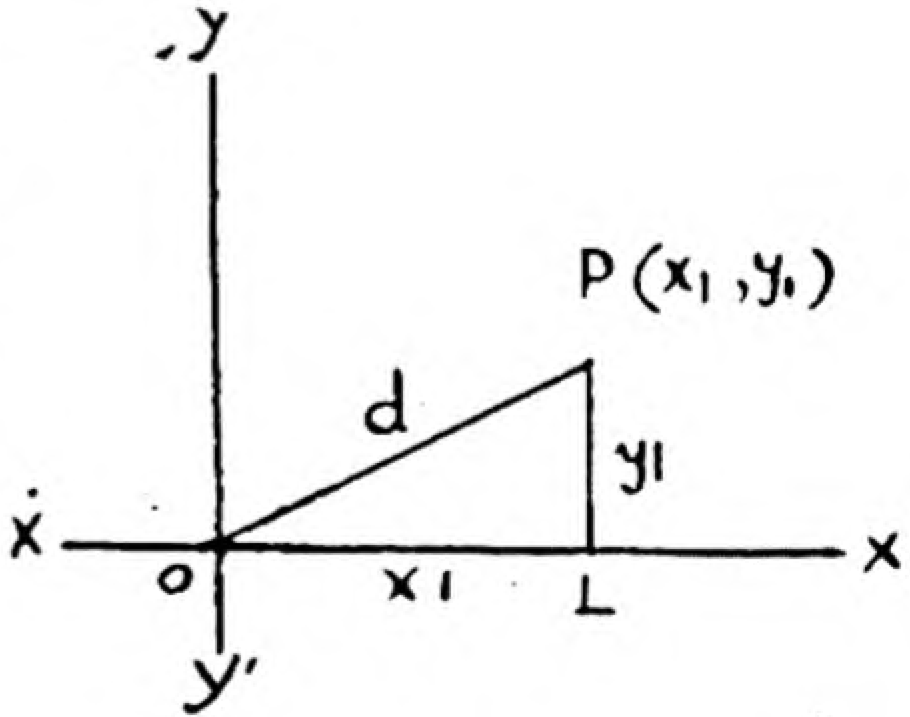
Here we have to find the distance between the points P(x_1, y_1) and the origin "O" whose co-ordinates are (0, 0). Complete the figure as shown. Then from the right-angled triangle POL, we have

$$OP^2=OL^2+PL^2 \quad \text{...(}i\text{)}$$

But OL $=x_1$ and PL $=y_1$

∴ (1) gives

$$d=\sqrt{x_1^2+y_1^2}.$$



Solved Examples

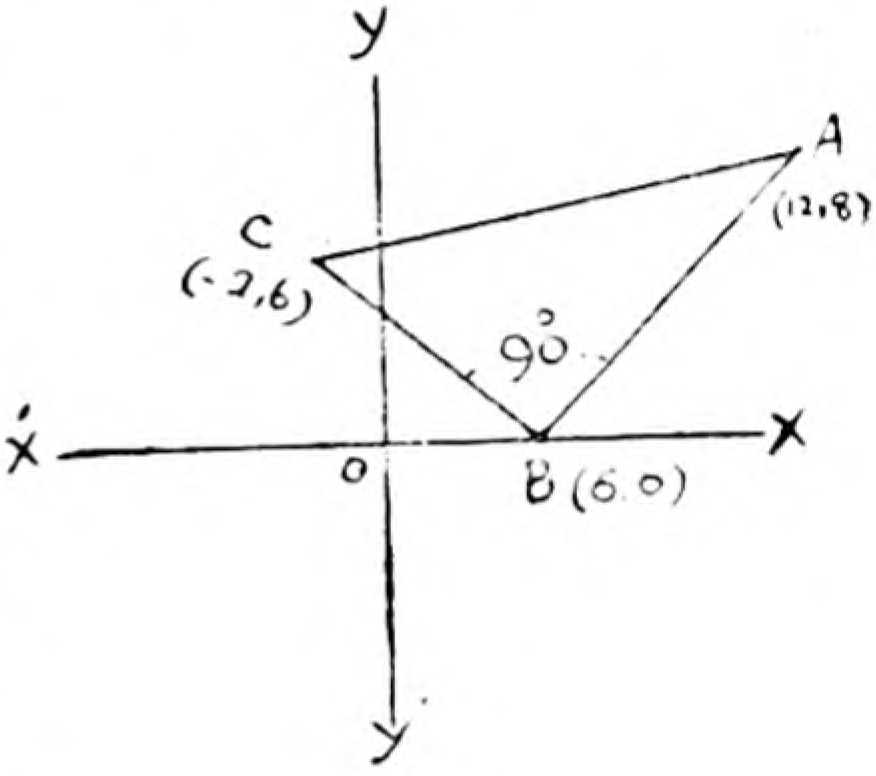
Ex. 1. Find the distance between the points (1, —2) and (—2, 2).

Sol. Here we have to take $(1, -2)$ in place of (x_1, y_1) and $(-2, 2)$ in place of (x_2, y_2).

$\therefore$ $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ gives $d=\sqrt{(-2-1)^2+(2+2)^2}$
$=\sqrt{9+16}=5.$

Ex. 2. Show that the points (12, 8), (−2, 6), and (6, 0) are the vertices of a right-angled triangle.

Sol. Let A, B, C be the points (12, 8), (6, 0) and (−2, 6) respectively.



Plot the points A(12, 8), B(6, 0), and C(−2, 6) carefully, on a *graph paper if possible.*

Now $AC^2=(12+2)^2+(8-6)^2$
$=200$
$AB^2=(12-6)^2+(8-0)^2$
$=100$
and $BC=(6+2)^2+(0-6)^2$
$=100$

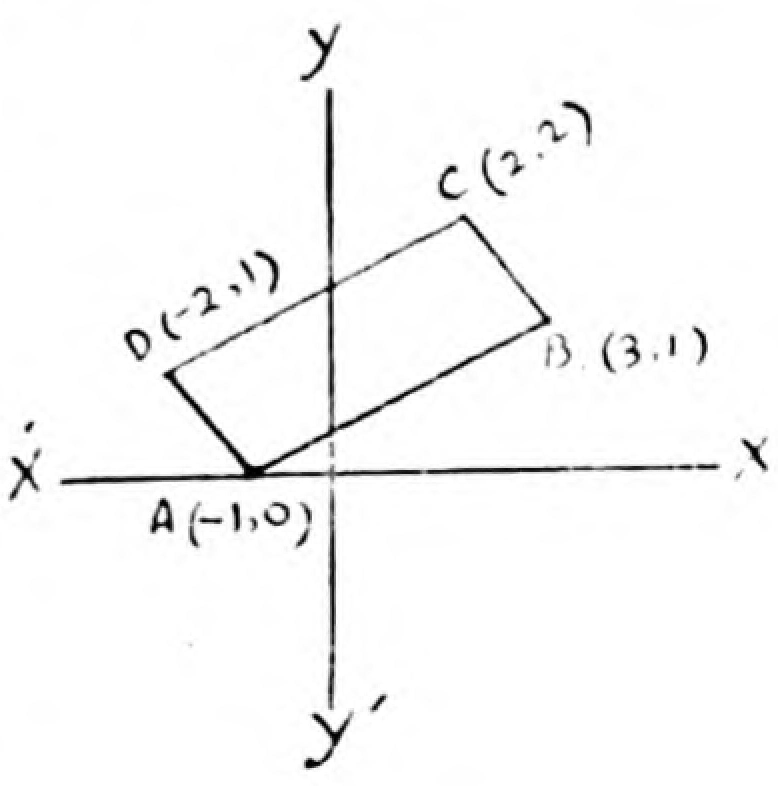
$\therefore$ $AC^2=AB^2+BC^2.$

Hence, by the converse of *Pythagoras* theorem, the triangle ABC is a right-angled triangle with ∠B=90°.

Note : *In questions of the type given above, it is advisable to plot the points on a graph paper. By doing so, the student will get a very clear figure.*

Ex. 3. Prove that the points (−1, 0), (3, 1), (2, 2), and (−2, 1) are the corners of a parallelogram.

Sol. Let A, B, C, D be the points (−1, 0), (3, 1), (2, 2), and −2, 1) respectively.

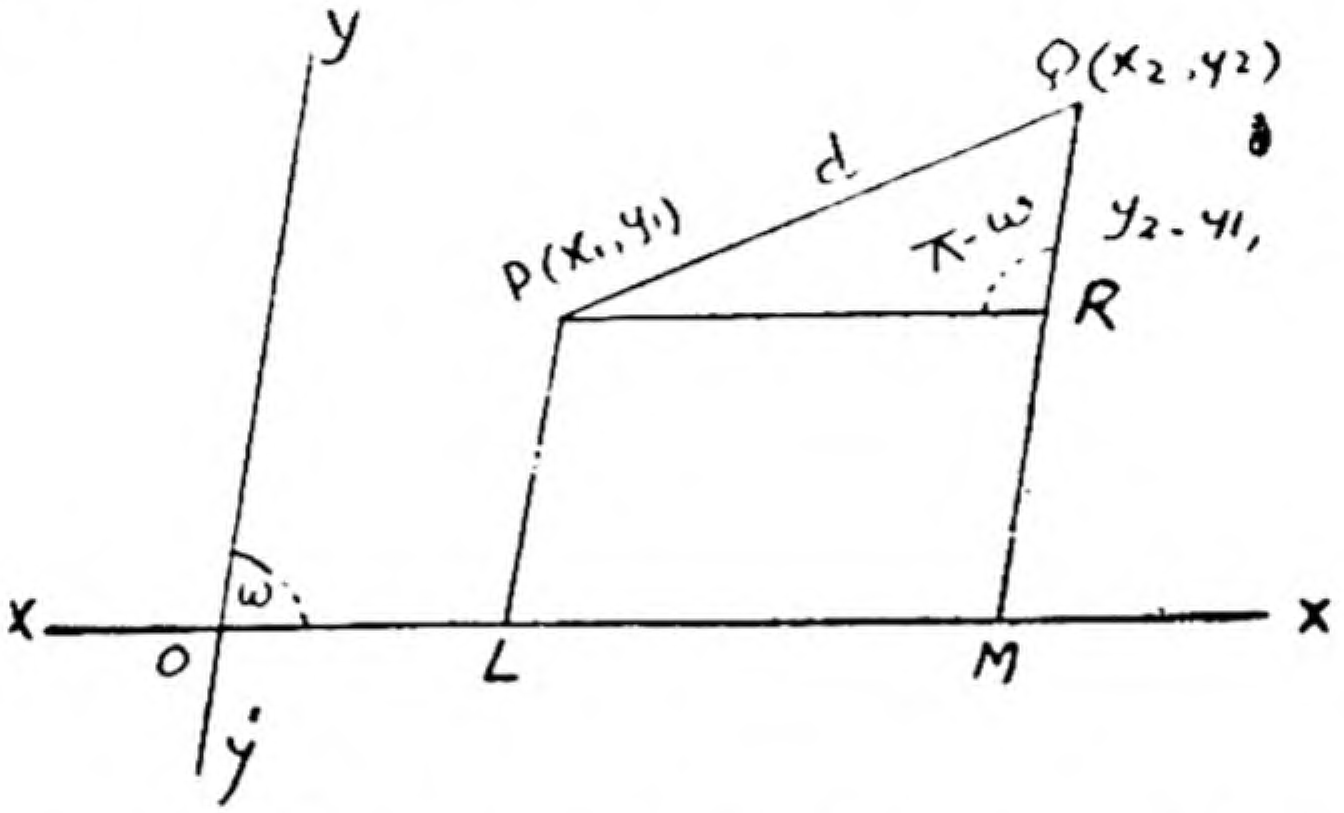


Plot these points as shown in the figure.

Now
$AB=\sqrt{(3+1)^2+(1-0)^2}=\sqrt{17}$
$BC=\sqrt{(3-2)^2+(1-2)^2}=\sqrt{2}$
$CD=\sqrt{(2+2)^2+(2-1)^2}=\sqrt{17}$
$DA=\sqrt{(-1+2)^2+(0-1)^2}=\sqrt{2}$

This shows that AB=CD, BC=DA which are the opposite sides of the quadrilateral ABCD. Hence the quadrilateral is a parallelogram.

1.8.* Distance between two points when the axes are oblique.



Let P (x_1, y_1) and Q (x_2, y_2) be the two points and let ∠XOY=ω.

Draw PL and QM parallel to OY meeting OX in L and M respectively.

Draw RR parallel to OX meeting QM in R.

Now $\quad PR=LM=OM-OL=x_2-x_1$

$\qquad QR=QM-RM=QM-PL=y_2-y_1$

and $\angle PRQ=\pi-\omega$

Let PQ=d ; then from the triangle PQR, we have

$$\cos(\pi-\omega)=\frac{PR^2+QR^2-PQ^2}{2.PR.QR} \quad \text{...(By Cosine formula)}$$

or $\quad PQ^2=PR^2+QR^2-2.PR.QR\cos(\pi-\omega)$

$\qquad =PR^2+QR^2+2.PR\,QR\cos\omega$

$\qquad\qquad [\because \cos(\pi-\omega)=-\cos\omega]$

or $\quad d^2=(x_2-x_1)^2+(y_2-y_1)^2+2(x_2-x_1)(y_2-y_1)\cos\omega$

$$\therefore \quad \mathbf{d}=\sqrt{(\mathbf{x}_2-\mathbf{x}_1)^2+(\mathbf{y}_2-\mathbf{y}_1)^2+\mathbf{2}(\mathbf{x}_2-\mathbf{x}_1)(\mathbf{y}_2-\mathbf{y}_1)\cos\omega}$$

Cor. If ω=90°, then the axes become *rectangulur*, and,

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad (\because \cos 90°=0)$$

which is the *same* formula as obtained under Article 1.7.

Note* : *Theorem coming under oblique axes are not included in the syllabus prescribed for the T. D. C. (Part I) students of the J. & K. University. These are, however, given here for the advantage of ambitious students.*

Exercise 1

Find the distance between the following pairs of points :—

1. $(2, 3), (5, 7)$
2. $(4, -7), (-1, 5)$
3. $(a, 0), (0, b)$
4. $(b+c, c+a), (c+a, a+b)$
5. $(a \cos \alpha, a \sin \alpha), (a \cos \beta, a \sin \beta)$
6. $(am_1^2, 2am_1), (am_2^2, 2am_2)$

7. For what value of x, the distance between the points $(x, 2)$ and $(3, 4)$ is 8 ?

8. A line is of length 10, and one end of it is at the point $(2, -3)$. If the abscissa of the other end be 10, prove that the ordinate of this other end must be either 3 or -9.

9. Show that the points $(a, a), (-a, a)$ and $(-a\sqrt{3}, a\sqrt{3})$ are the vertices of an equilateral triangle.

10. Show that the points $(0, -1), (2, 1), (0, 3)$ and $(-2, 1)$ are the corners of a square.

11. Show that the points $(1, 1), (-\frac{1}{2}, \frac{1}{2})$ and $(1, 2)$ form a right-angled triangle.

12. Show that the points $(2, -1), (4, -1), (-1, -2)$, and $(1, -4)$ form the angular points of a parallelogram.

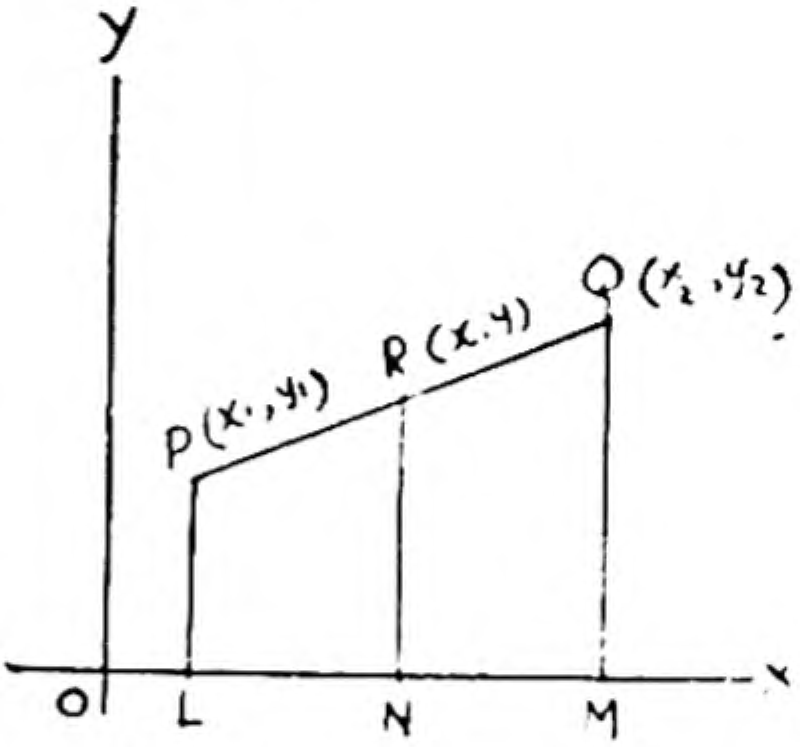
13. The point (x, y) is equidistant from the points $(-1, 1)$ and $(3, -2)$. Show that $8x-6y+11=0$.

✓ Imp

1.9. Section Formulae

To find the co-ordinates of the point which divides the join of (x_1, y_1) *and* (x_2, y_2) *in the ratio* $l : m$.

(i) Internal Division



Let P (x_1, y_1) and Q (x_2, y_2) be the two given points and R(x, y) be the point which divides PQ in the ratio $l : m$ *internally*, so that

$$\frac{PR}{RQ}=\frac{l}{m}$$

Draw PL, QM and RN perpendiculars on x-axis meeting it in L, M and N respectively. Then

$$\frac{LN}{NM}=\frac{PR}{RQ}=\frac{l}{m} \qquad \ldots(1)$$

Now $\quad LN=ON-OL=x-x_1$, and
$\quad NM=OM-ON=x_2-x$

$\therefore$ from (1) we get

$$\frac{x-x_1}{x_2-x}=\frac{l}{m}$$

or $\quad lx_2-lx=mx-mx_1$

or $\quad lx_2+mx_1=(l+m)x$

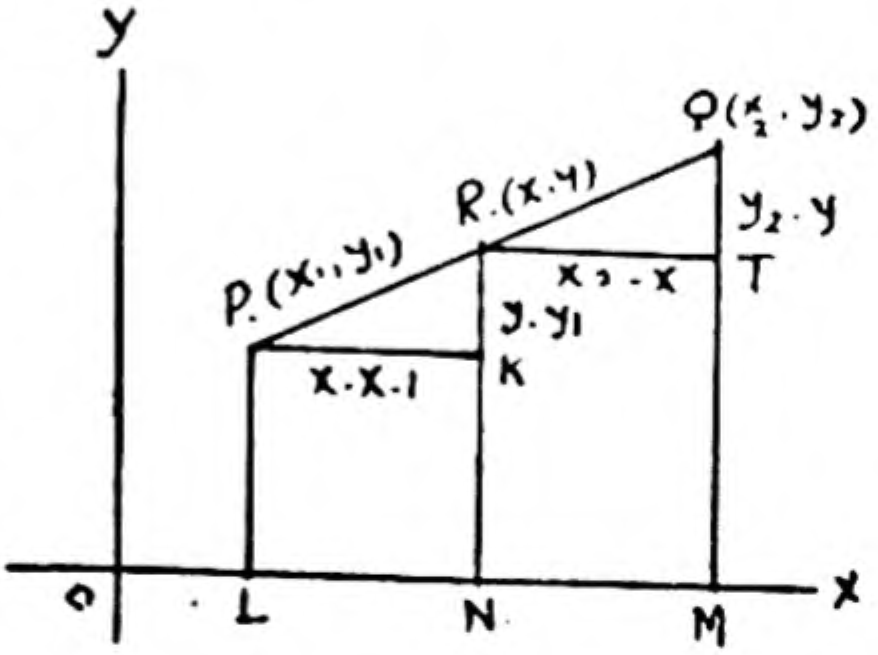
$$\therefore \qquad \mathbf{x=\frac{lx_2+mx_1}{l+m}}$$

Similarly $\mathbf{y=\dfrac{ly_2+my_1}{l+m}}$

Alternative Proof : (*to be preferred*)

Let P(x_1, y_1) and Q (x_2, y_2) be the points and R(x, y) the point which divides PQ *internally* in in the ratio $l : m$.

Draw PL, QM, RN perpendiculars on OX, PK and RT perpendiculars on RN and QM respectively.



Now the triangles PQR and RTQ are similar ; hence their corresponding sides being proportional, we have

$$\frac{PR}{RQ}=\frac{PK}{RT}=\frac{RK}{QT} \qquad \ldots(1)$$

But $\dfrac{PR}{RQ}=\dfrac{l}{m}$, $\dfrac{PK}{RT}=\dfrac{LN}{NM}=\dfrac{ON-OL}{OM-ON}=\dfrac{x-x_1}{x_2-x}$

and $\dfrac{RK}{QT}=\dfrac{RN-KN}{QM-TM}=\dfrac{RN-PL}{QM-RN}=\dfrac{y-y_1}{y_2-y}$

$\therefore$ from (1) we get

$$\frac{l}{m}=\frac{x-x_1}{x_2-x}=\frac{y-y_1}{y_2-y}$$

which gives us

$$x=\frac{lx_2+mx_1}{l+m}$$

$$y=\frac{ly_2+my_1}{l+m}$$

Cor. *If R be the mid-point of PQ, the ratio* $l:m$ *becomes* $1:1$. *Hence the co-ordinates of the mid-point are*

$$\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$$

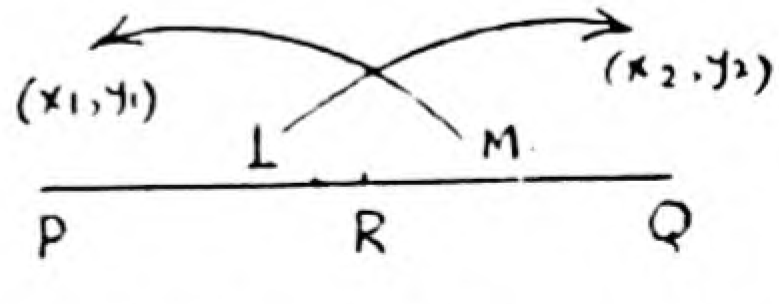
Note : *The following is the working rule for writing down the co-ordinates of the point which divides the join of the two given points internally in the ratio* $l:m$.

(*i*) *Multiply* l *by the* x*-co-ordinate of the point remote from* l, *and* m *by the* x*-co-ordinate of the point remote from* m.

(*ii*) *add these products, and*

(*iii*) *divide the sum by* $l+m$.

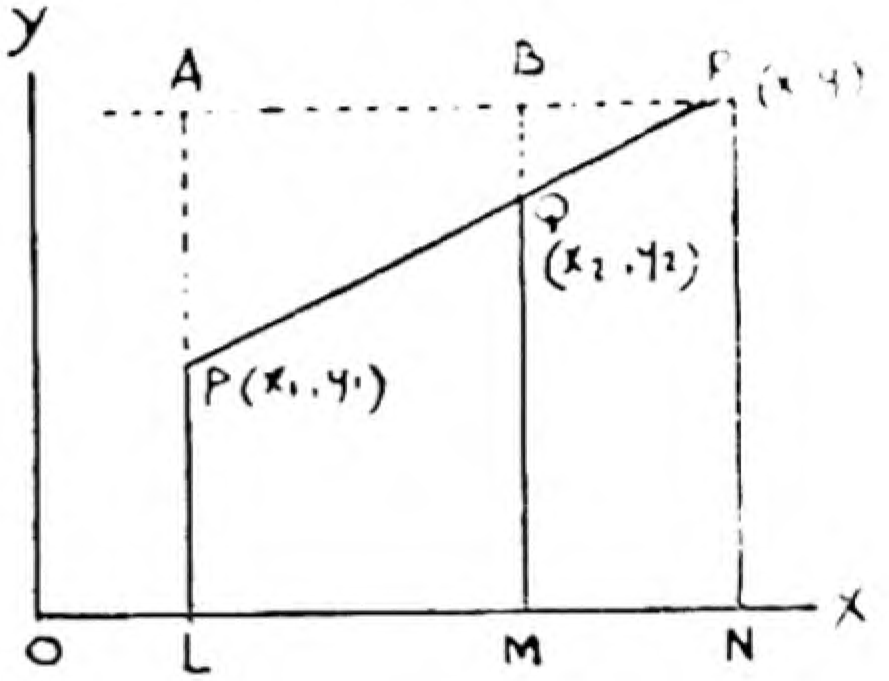
This will give the x-co-ordinate of the point of division.



The y-co ordinate of the point will also be got in the same manner.

(*ii*) **External Division.**

Let P$(x_1, y_1)$and Q(x_2, y_2) be the two points and R the point which divides PQ *externally* in the ratio $l : m$.



From P, Q and R draw PL, MQ and RN perpendiculars on OX. Through R, draw a straight line RBA parallel to the x-axis meeting LP and MQ produced in A and B respectively.

Then from the similar triangles ARP and BRQ, we have

$$\frac{AR}{BR}=\frac{PR}{RQ}=\frac{l}{m} \qquad \ldots(1)$$

But $AR=LN=ON-OL=x-x_1$

and $BR=MN=ON-OM=x-x_2$

$\therefore$ From (1), we have

$$\frac{x-x_1}{x-x_2}=\frac{l}{m}$$

or $lx-lx_2=mx-mx_1$

or $(l-m)x=lx_2-mx_1$

$$\therefore \qquad \mathbf{x=\frac{lx_2-mx_1}{l-m}}$$

$$\text{Again,} \quad \frac{AP}{BQ}=\frac{PR}{PQ}=\frac{l}{m} \qquad \ldots(2)$$

But $AP=AL-PL=RN-PL=y-y_1$

and $BQ=BM-QM=RN-QM=y-y_2$

$\therefore$ From (2), we have

$$\frac{y-y_1}{y-y_2}=\frac{l}{m}$$

or $ly-ly_2=my-my_1$

or $(l-m)y=ly_2-my_1$

$$\therefore \qquad \mathbf{y=\frac{ly_2-my_1}{l-m}}$$

Note : (1) *The following is the working rule for writing down the co-ordinates of the point which divides the join of the two given points externally in the ratio $l : m$.*

Change m into $-m$ and write down the co-ordinates of the point which divides the join of the given points internally in the ratio $l : -m$.

Thus, if $R(x, y)$ is such point, we have

$$x=\frac{lx_2+(-m)x_1}{l+(-m)}=\frac{lx_2-mx_1}{l-m}$$

and $$y=\frac{ly_2+(-m)y_1}{l+(-m)}=\frac{ly_2-my_1}{l-m}$$

Note : (2) If R divides PQ *internally* in the ratio $l : m$, then $\frac{PR}{RQ}=\frac{l}{m}$ and if it divides PQ *externally* in the said ratio, then $\frac{PR}{RQ}=-\frac{l}{m}$ for in this case PR and RQ will have *different* signs.

1.10. Some Important Theorems

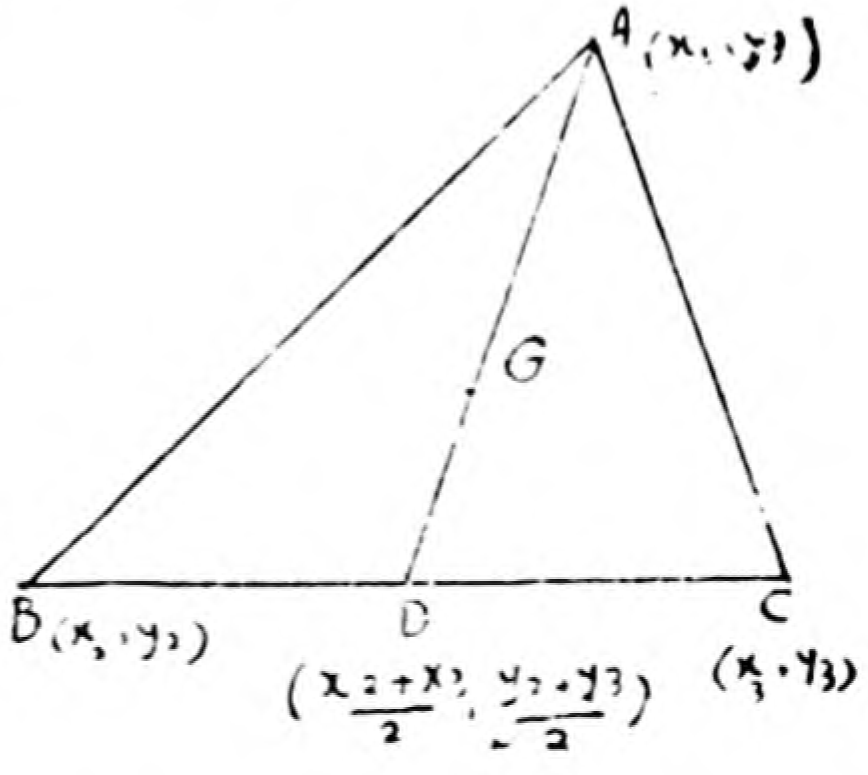
(*a*) *To prove that the medians of a triangle are concurrent.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC. Let D be the mid-point of BC.

$\therefore$ The co-ordinates of D are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

Let G divide in the ratio 2 : 1,

Then the co-ordinates of G are given by :—



$$x=\frac{1.x_1+2.\frac{x_2+x_3}{2}}{1+2}=\frac{x_1+x_2+x_3}{3}$$

and $$y = \frac{1.y_1 + 2.\dfrac{y_2+y_3}{2}}{1+2} = \frac{y_1+y_2+y_3}{3}$$

The symmetry of the result shows that we can come to the same point if the other two medians are also treated in a similar manner. Hence the three medians are concurrent.

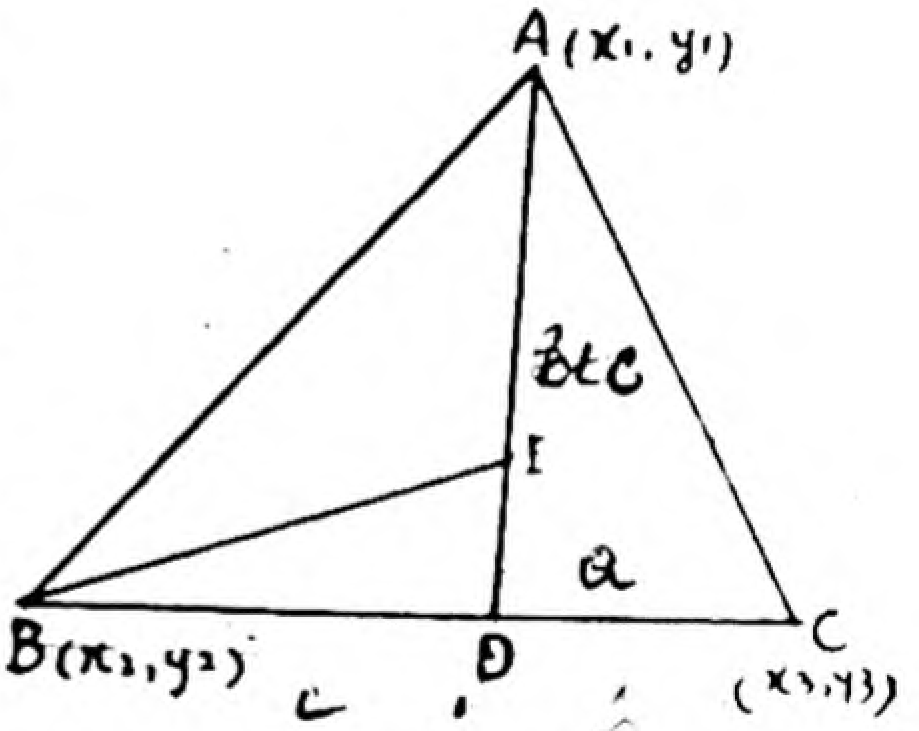
Note: (*i*) The point of *concurrence* of the three medians is called the **centroid** of the triangle, and its co-ordinates, as shown above, are:

$$\frac{x_1+x_2+x_3}{3}, \quad \frac{y_1+y_2+y_3}{3}$$

(*ii*) The centroid divides each *median* in the ratio **2 : 1.**

(*b*) *To prove that the bisectors of the angles of a triangle are concurrent.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC. Let AD be the *internal* bisector of angle A meeting BC in D. Then D divides BC in the ratio AB : AC *i.e.*, $c : b$.



Let the *internal* bisector of angle B meet AD in I. Then I is defined as the **in-centre** of the triangle.

Now $$\frac{BD}{DC} = \frac{c}{b}$$

∴ the co-ordinates of D are given by

$$x = \frac{bx_2+cx_3}{b+c}, \; y = \frac{by_2+cy_2}{b+c}$$

Again $$\frac{BD}{DC} = \frac{c}{b}$$

or $$\frac{BD}{BD+DC} = \frac{c}{c+b}$$ (*Please note this step*)

$$\text{or}\quad \frac{\mathrm{BD}}{\mathrm{BC}}=\frac{c}{c+b}$$

$$\text{or}\quad \frac{\mathrm{BD}}{a}=\frac{c}{b+c}\qquad \therefore\ \mathrm{BD}=\frac{ca}{b+c}\qquad \ldots(i)$$

Now BI, the *internal* bisector of angle B, meets AD in I

$$\therefore\quad \frac{\mathrm{ID}}{\mathrm{IA}}=\frac{\mathrm{BD}}{\mathrm{AB}}=\frac{\frac{ca}{b+c}}{c}\qquad \text{[From } (i)\text{]}$$

$$=\frac{a}{b+c}$$

$\therefore$ the co-ordinates of I are given by

$$x=\frac{a\times x_1+(b+c)\times\dfrac{bx_2+cx_3}{b+c}}{a+(b+c)}=\frac{\mathbf{ax_1+bx_2+cx_3}}{\mathbf{a+b+c}}$$

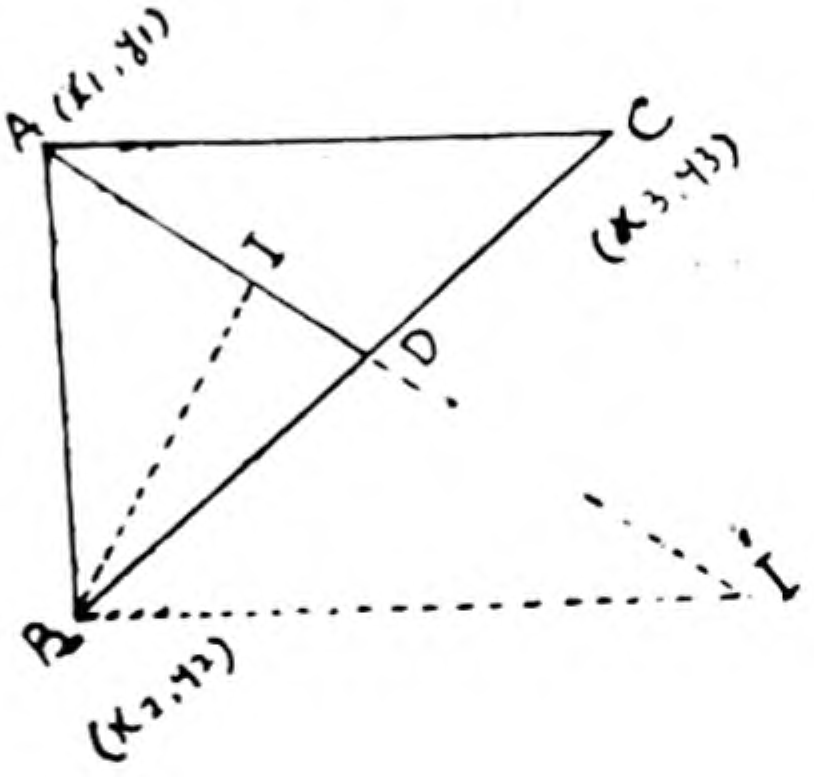
$$\text{and}\quad y=\frac{a\times y_1+(b+c)\times\dfrac{by_2+cy_3}{b+c}}{a+(b+c)}=\frac{\mathbf{ay_1+by_2+cy_3}}{\mathbf{a+b+c}}$$

The result being symmetrical, we conclude that the bisectors of the angles of a triangle are concurrent.

***1.11.** *To show that the co-ordinates of the centre of the escribed circle opposite to the angle A are given by*

$$x=\frac{-ax_1+bx_2+cx_3}{a+b+c}\ \text{and}\ y=\frac{-ay_1+by_2+cy_3}{-a+b+c}$$

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the co-ordinates of the vertices of the triangle ABC.



Let BI_1, the bisector of the exterior angle ABC meet AD, the bisector of angle A, produced in I_1. Then I_1 is defined as the centre of the *escribed* circle opposite to the angle A, and is generally known as *e*-centre.

Now,

$$\frac{AI_1}{I_1D}=\frac{AB}{BD}=\frac{b+c}{a}$$

Therefore, I_1 divides AD *externally* in the ratio $b+c : a$. Hence its co-ordinates are given by

$$x=\frac{(b+c)\times\frac{bx_2+cx_3}{b+c}-ax_1}{(b+c)-a}=\frac{-ax_1+bx_2+cx_3}{-a+b+c}$$

and $$y=\frac{(b+c)\times\frac{by_2+cy_3}{b+c}-ay_1}{(b+c)-a}=\frac{-ay_1+by_2+cy_3}{-a+b+c}$$

We can in *a like manner* show that the co-ordinates of the centres of the escribed circles opposite to angles B and C are

$$\frac{\mathbf{ax_1-bx_2-cx_3}}{\mathbf{a-b+c}},\quad\frac{\mathbf{ay_1-by_2+cy_3}}{\mathbf{a-b+c}}$$

and $$\frac{\mathbf{ax_1+bx_2-cx_3}}{\mathbf{a+b-c}},\quad\frac{\mathbf{ay_1+by_2-cy_3}}{a+b-c}.$$

Note* : *This theorem may be omitted by less ambitious students*.

Solved Examples

Ex. 1. Find the co-ordinates of the points which divide the the join of (2, 5) and (4, 7) *internally* and *externally* in the ratio 3 : 5

Sol. Here $x_1=2$, $x_2=4$; $y_1=5$, $y_2=7$; $l=3$, $m=5$.

(*i*) ∴ the co-ordinates of the point of *internal* division are given by

$$x=\frac{lx_2+mx_1}{l+m}=\frac{3\times4+5\times2}{3+5}=\frac{22}{8}=\frac{11}{4}$$

$$y=\frac{ly_2+my_1}{l+m}=\frac{3\times7+5\times5}{3+5}=\frac{46}{8}=\frac{23}{4}$$

(*ii*) The co-ordinates of the point of *external* division are given by

$$x = \frac{lx_2 - mx_1}{l-m} = \frac{3\times4 - 5\times2}{3-5} = \frac{2}{-2} = -1$$

$$y = \frac{ly_2 - my_1}{l-m} = \frac{3\times7 - 5\times5}{3-5} = \frac{-4}{-2} = 2$$

Ex. 2. *Find the ratio in which the x-axis divides the join of the points P(4, 6) and Q(3, −8). What are the co-ordinates of the point of division ?*

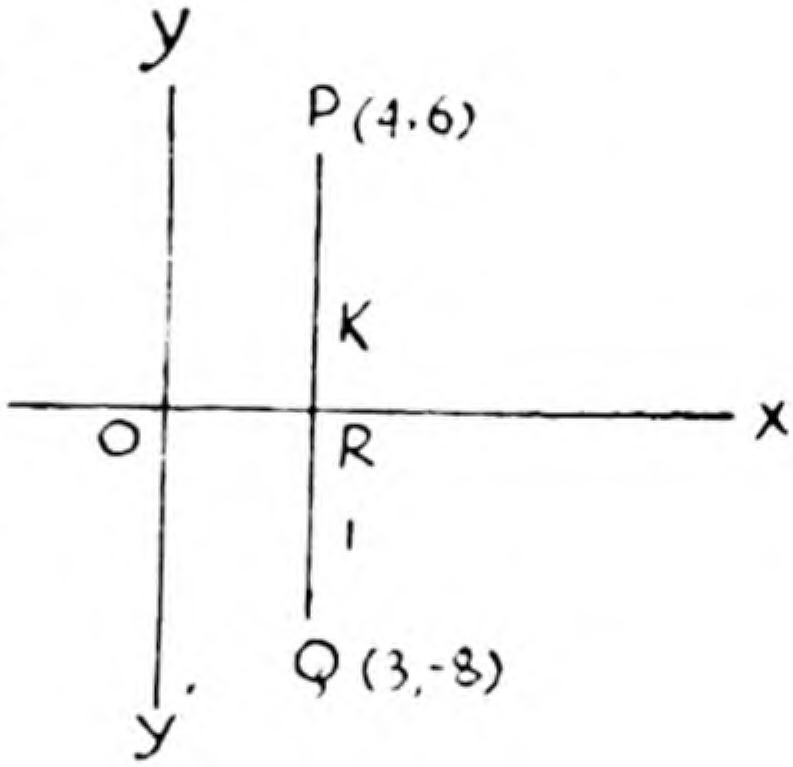
Sol. Let R be the point in which the join of PQ is cut by the x-axis and k : 1 be the ratio in which R divides PQ. Evidently, the y co-ordinate of R is zero.

$\therefore$ Using $\mathfrak{z} = \dfrac{ly_2 + my_1}{l+m}$,

we get,

$$0 = \frac{-8k+6}{k+1} \text{ which gives}$$

$$k = \frac{3}{4}$$



Hence the required ratio is $\frac{3}{4} : 1$ or $3 : 4$

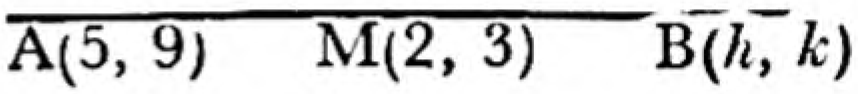
Next, from $x = \dfrac{lx_2 + mx_1}{l+m}$, we get,

$$x = \frac{3\times3 + 4\times4}{3+4} = \frac{25}{7}$$

Hence the co-ordinates of R are $\left(\frac{25}{7}, 0\right)$.

Note : *The stndent should note that by supposing the ratio to be l : m in place of k : 1 in the above example, it would have been difficult for us to find the co-ordinates of R.*

Ex. 3. The mid-point of a line-segment is (2, 3) and one end is (5, 9) ; find the co-ordinates of the other end.



Sol. Let A (5, 9) be one end and B(h, k) the other end. Let M(2, 3) be the mid-point of AB. Then from $x=\frac{x_1+x_2}{2}$ and $y=\frac{y_1+y_2}{2}$, we get

$$2=\frac{5+h}{2} \text{ and } 3=\frac{9+k}{2}$$

These give $h=-1$ and $k=-3$

Hence the co-ordinates of B are $(-1, -3)$.

Ex. 4. Find the centroid of the triangle the co-ordinates of whose vertices are : (0, 1), (−1, 2), (−2, −3)

Sol. Here $x_1=0$, $x_2=-1$. $x_3=-2$
$y_1=1$, $y_2=2$, $y_3=-3$

Therefore, from the formula $x=\frac{x_1+x_2+x_3}{3}$ and $y=\frac{y_1+y_2+y_3}{3}$, we have

$$x=\frac{0-1-2}{3}=-1 \text{ and } y=\frac{1+2-3}{3}=0$$

Hence $(-1, 0)$ are the co-ordinates of the centroid.

1.12. Choice of axes for proving some Geometrical Theorems by Co-ordinate Geometry (*i.e.*, Analytically)

Rule I. *For problems relating to two given perpendicular lines, take these lines as axes, as explained below :—*

(i) For problems relating to a right-angled triangle ABC (right-angled at B) take BC and BA as the axes of x and y respectively, and B as the origin.

(*ii*) *For problems relating to a rectangle or a square ABCD, take AB and AD as these axes, and A as the origin.*

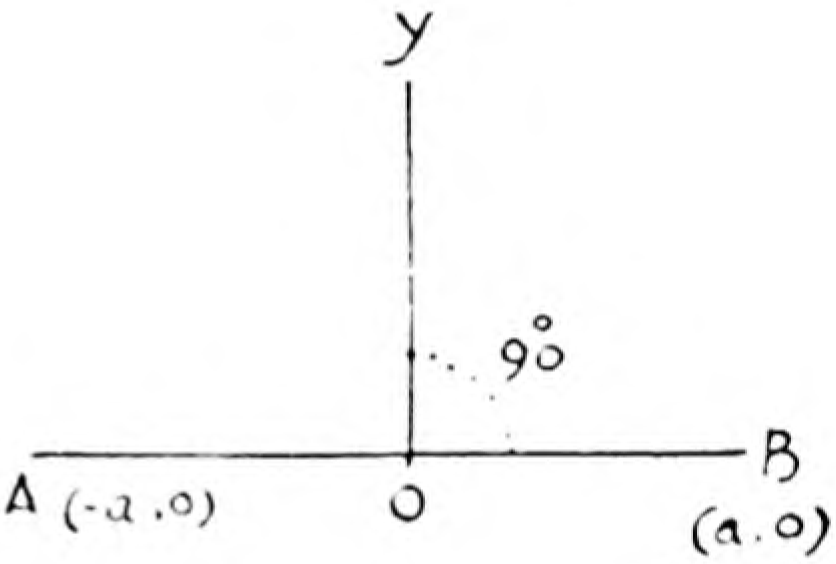
Note : Sometimes, for the sake of convenience we take the centre of a rectangle or of a square as the origin, and lines through it, parallel to the sides, as the axes.

Rule II. *For problems relating to two given points A and B, take O, the mid-point of AB, as the origin, and OB as the x-axis, so that OY, the line through O perpendicular to OB, may be y-axis.*

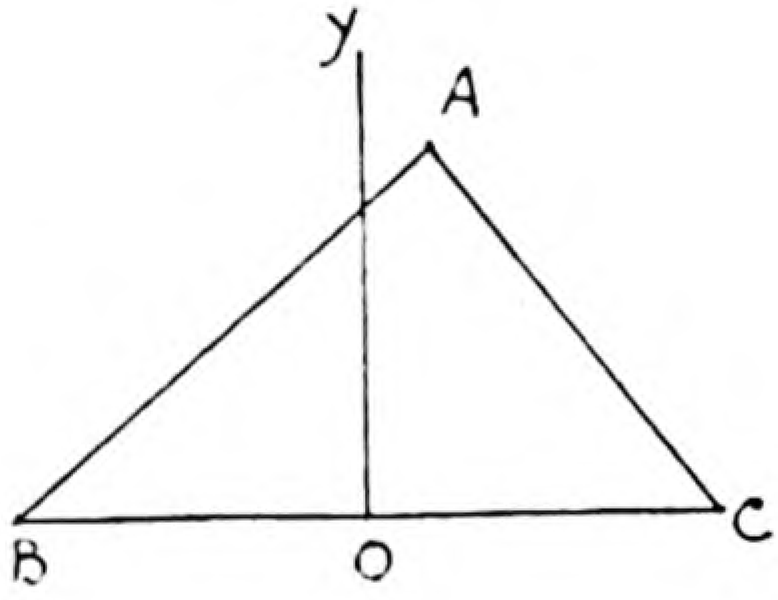
Explanation :—

Let AB $=2a$

Then B is $(a, 0)$ and A is $(-a, 0)$.



Cor. *For problems relating to a triangle ABC, take O, the mid-point of BC, as the origin, and OC as the x-axis, so that OY, the line through O perpendicular OC, may be the y-axis.*



Explanation :—

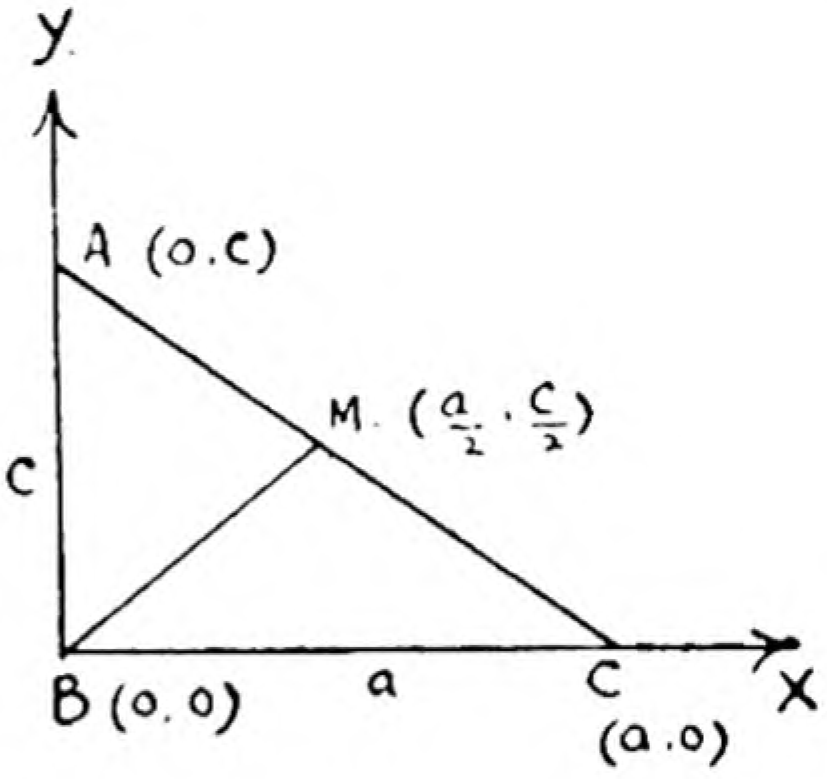
Let BC $=2a$

Then C is $(a, 0)$ and B is $(-a, 0)$. Let A be (x_1, y_1).

Solved Examples

Ex. 1. *Prove, analytically that in a right-angled triangle, the poid-point of the hypotenuse is equidistant from the three angular mints.* **(Very Important)**

Sol. Let ABC be the right-angled traingle right-angled at B.



Take BC and BA as the axis of x and y respectively. Put BC$=a$ and BA $=c$, then the co-ordinates of M, the mid-point of AC, are $\left(\frac{a}{2}, \frac{c}{2}\right)$

$$\therefore \quad AM = \sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{c}{2}-c\right)^2} = \sqrt{\frac{a^2+c^2}{2}}$$

$$CM = \sqrt{\left(\frac{a}{2}-a\right)^2+\left(\frac{c}{2}-0\right)^2} = \sqrt{\frac{a^2+c^2}{2}}$$

$$\text{and } BM = \sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{c}{2}-0\right)^2} = \sqrt{\frac{a^2+c^2}{2}}$$

$$\therefore \; AM=CM=RM$$

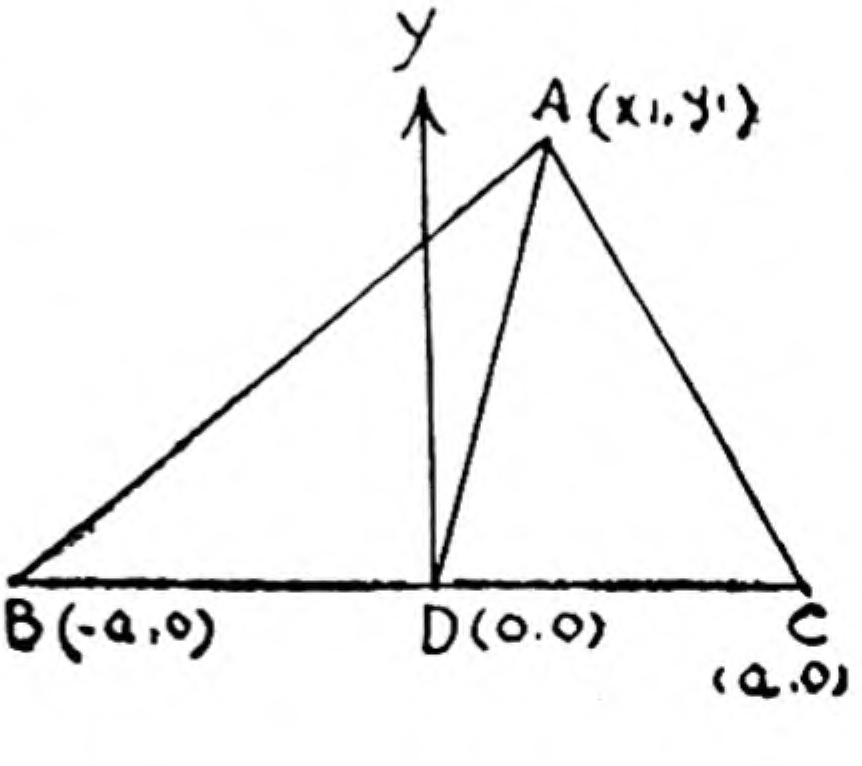
Hence M is equidistant from A, C, and B.

Ex. 2. In any triangle ABC, prove that

$$AB^2+AC^2=2(AD^2+DC^2)$$

where D is the mid-point of BC.

Sol. Take D as the origin, DC as the axis of x, and DY, the line through D perpendicular to BC, as the axis of y. Take the co-ordinates of C as $(a, 0)$ and those of B as $(-a, 0)$. Let (x_1, y_1) be the co-ordinates of A.



Now that the co-ordinates of D are (0, 0), we have

$$\begin{aligned}&2(AD^2+DC^2)\\ &\quad =2[\{(x_1-0)^2+(y_1-0)^2\}+\{(a-0)^2+(0-0)^2\}]\\ &\quad =2(x_1^2+y_1^2+a^2)\end{aligned}$$

$$\text{Also } AB^2+AC^2=\{(x_1+a)^2+(y_1-0)^2\}+\{(x_1-a)^2+(y_1-0)^2\}$$
$$=2x_1^2+2y_1^2+2a^2=2(x_1^2+y_1^2+a^2)$$

Hence
$AB^2+AC^2=2(AD^2+DC^2)$.

Exercise 2

Find the co-ordinates of the point which divides the line joining :—

1. The points (1, 3), (2, 7) *internally* in the ratio 3 : 4.

2. The points (1, 3), (2, 7) *externally* in the ratio 3 : 4.

3. The points (−1, 2), (4, −5) *internally* as well as *externally* in the ratio 2 : 3.

4. Joining the points (−3, −4), (−8, 7) in the ratio 7 : 5 externally as well as internally.

5. The line joining the points (1, −2) and (−3, 4) is trisected ; find the co-ordinates of the points of trisection.

6. The line joining the points (−6, 8) and (8, −6) is divided into four equal parts. Find the co-ordinates of the points of section.

(7, 1)

7. The point (0, 2) is the mid-point of the line joining A(−3, −4) and B find the co-ordinates of B.

8. The mid-points of the sides of a triangle are (2, 1), (−6, 3), and (−4, −5) ; find the co-ordinates of the vertices of the triangle.

9. Find the centroid of the triangle whose vertices are (3, −5), (−7, 4), and (10, −2).

10. The centroid of a triangle ABC is $(\frac{5}{3}, -\frac{10}{3})$ and the co-ordinates of A and B are (2, 3) and (7, −5) respectively. Find the co-ordinates of C.

11. In what ratio is the line joining the points (4, 5) and (1, 2) divided by.

12. Find the lengths of the medians of the triangle whose vertices are :—

(*i*) (3, 5), (5, 3), (7, 7). (*ii*) (7, 1), (3, 9), (−4, −6).

13. Show that the lines joining (3, 5), (7, 7) and (1, 4), (9, 8) bisect each other.

14. The co-ordinates of the three vertices of parallelogram are (−4, −2), (2, 0), and (8, 6). Find the co-ordinates of the fourth corner opposite to the corner (2, 0).

15. Find the co-ordinates of the in-centre of the triangles formed by the points :—

(*i*) (4, −2), (−2, 4), (5, 5)

(*ii*) $(-3, -2), (5, 0), (-\frac{1}{3}, \frac{13}{3})$.

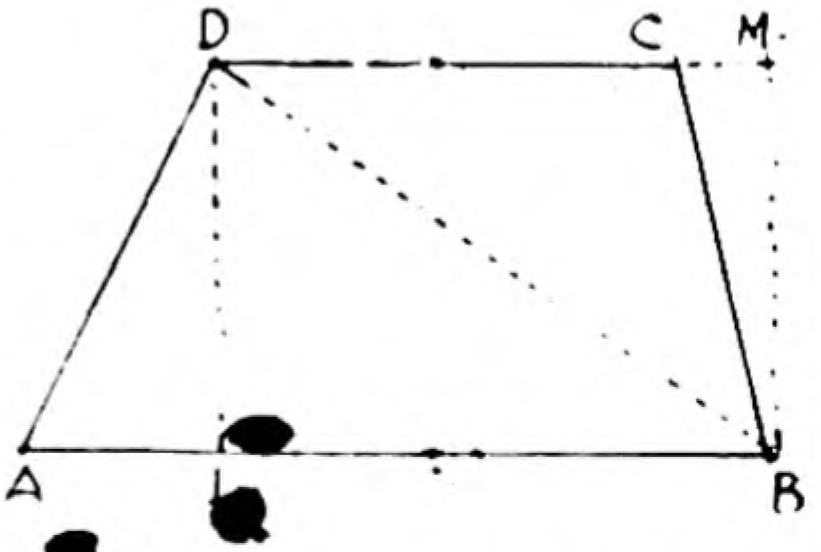
16. If G is the centroid of a triangle ABC and O any point, show that :—

(*i*) $3(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2$

(*ii*) $OA^2+OB^2+OC^2=GA^2+GB^2+GC^2+3GO^2$.

1.13. Area of a trapezium.

Let ABCD be the trapezium with AB and CD as its parallel sides. Join BD, and draw DL and BM perpendiculars on AB and DC (produced) respectively.



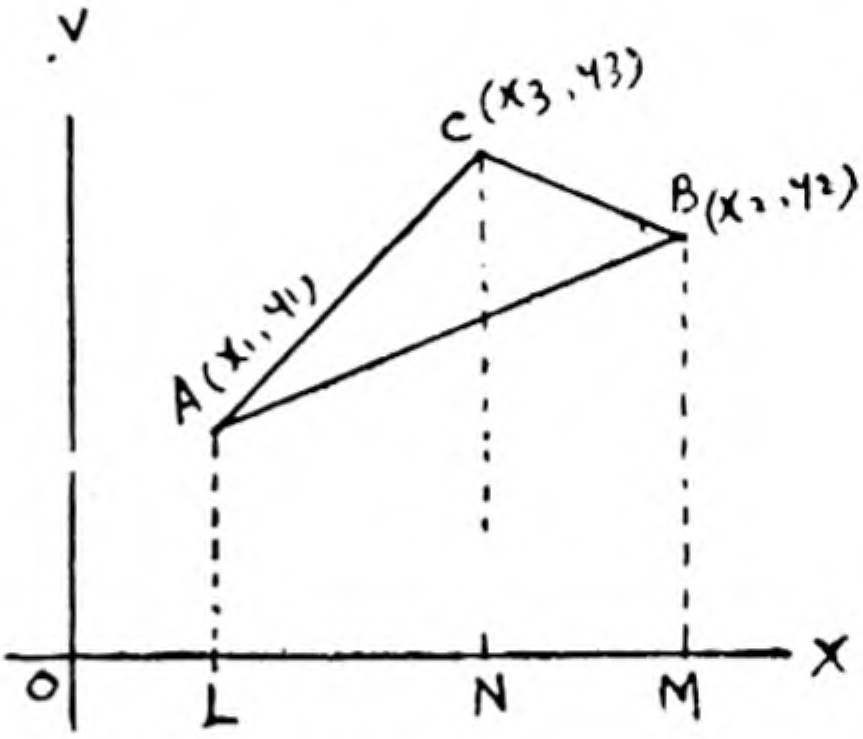
∴ the area ABCD
=area of △ ABD + area of △ BCD
$=\frac{1}{2}.AB.DL+\frac{1}{2}.DC.BM$
$=\frac{1}{2}\times(AB+DC)\times DL$ [∵ DL=BM]

Hence the area of a trapezium (i.e., a quadrilaterat having two sides parallel) is equal to one half the sum of the two parallel sides multiplied by the perpendicular distance between them.

Imp.

1.14. To find the area of a triangle in terms of the co-ordinates of its vertices.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of the triangle. Draw AL, BM, and CN perpendiculars on OX. Let Δ denote the area of the triangle.



Then

$$\Delta = \text{Trapezium ALNC} + \text{trapezium CNMB} - \text{trapezium ALMB}$$

$$= \tfrac{1}{2}(AL + CN) \times LN + \tfrac{1}{2}(CN + MB) \times MN - \tfrac{1}{2}(AL + MB) \times LM \qquad (\textit{Article 1.14})$$

$$= \tfrac{1}{2}[(y_1 + y_3)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) - (y_1 + y_2)(x_2 - x_1)]$$

$$= \tfrac{1}{2}[(\mathbf{x}_1\mathbf{y}_2 - \mathbf{x}_2\mathbf{y}_1) + (\mathbf{x}_2\mathbf{y}_3 - \mathbf{x}_3\mathbf{y}_2) + (\mathbf{x}_3\mathbf{y}_1 - \mathbf{x}_1\mathbf{y}_3)] \qquad \ldots(1)$$

This can be written even as under :—

$$\Delta = \tfrac{1}{2}[\mathbf{x}_1(\mathbf{y}_2 - \mathbf{y}_3) + \mathbf{x}_2(\mathbf{y}_3 - \mathbf{y}_1) + \mathbf{x}_3(\mathbf{y}_1 - \mathbf{y}_2)] \qquad \ldots(2)$$

If we use the determinant notation, the above result can be put as :—

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot 3$$

Cor. *The area of the triangle whose vertices are the origin and the points (x_1, y_1) and (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$.*

Note : *The student is advised to memorize the result (2) in preference to result (1). He is also advised to understand the following rule in this connection :—*

(*1*) *Write down* $\frac{1}{2}$ *outside the long bracket.*

(*2*) *Inside this long bracket, write down* $x_1(y_2-y_3)$ *as shown below :—*

$$\Delta=\tfrac{1}{2}[\mathbf{y}_1(\mathbf{y}_2-\mathbf{x}_3)+\qquad+\qquad \quad \ldots(4)$$

(*3*) *Change* x_1 *to* x_2, y_2 *to* y_3, *and* y_3 *to* y_1 *to get* $x_2(y_3-y_1)$ *and fill up the first gap within the long bracket by inserting in it* $x_2(y_3-y_1)$.

(*4*) *Change* x_2 *to* x_3, y_3 *to* y_1, *and* y_1 *to* y_2 *in* $x_2(y_3-y_1)$ *to get* $x_3(y_1-y_2)$, *and fill up the second gap by it within the long bracket in* (*4*) *above.*

(*5*) *The complete area formula will thus be got as under :—*

$$\Delta=\tfrac{1}{2}[\mathbf{x}_1(\mathbf{y}_2-\mathbf{y}_3)+\mathbf{x}_2(\mathbf{y}_3-\mathbf{y}_1)+\mathbf{x}_3(\mathbf{y}_1-\mathbf{y}_2)].$$

Note : (2) *The area of the triangle always means the area in magnitude. Therefore, if in a numerical example we get a negative result, we change its sign to get the area in magnitude.*

1.15. Condition for the collinearity of three points.

If A(x_1, y_1), B(x_2, y_2), and C(x_3, y_3) lie in a straight line, then the area of the triangle formed by these three points is zero. Thus

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0 \qquad [\because \tfrac{1}{2}\neq 0]$$

which gives the required condition.

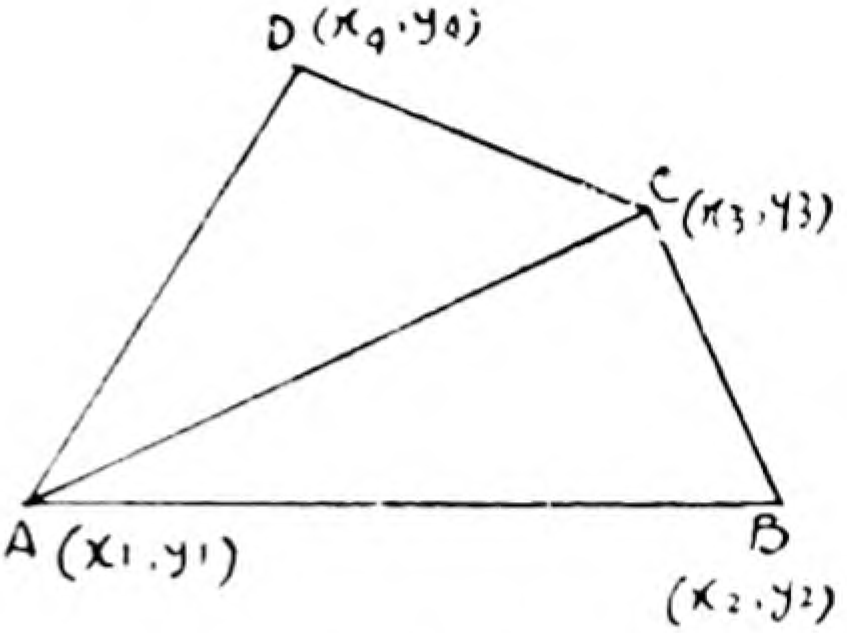
If we make use of the determinant notation this condition can be put as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Note : *Other tests for the collinearity of three points will be discussed in Chapter III* (*Article 3.16*).

1.16. *To find the area of a quadrilateral whose angular points are* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, *and* $D(x_4, y_4)$.

Draw AC the diagonal of the quadrilateral. Then the quadrilateral



$$\begin{aligned}\text{ABCD} &= \triangle\text{ABC} + \triangle\text{ACD}\\ &= \tfrac{1}{2}[x_1(y_2 - y_3)\\ &\quad + x_2(y_3 - y_1)\\ &\quad + x_3(y_1 - y_2)] +\\ &\quad \tfrac{1}{2}[x_1(y_3 - y_4)\\ &\quad + x_3(y_4 - y_1)\\ &\quad + x_4(y_1 - y_2)]\end{aligned}$$

$$= \tfrac{1}{2}[\mathbf{x}_1\mathbf{y}_2 - \mathbf{x}_2\mathbf{y}_1) + (\mathbf{x}_2\mathbf{y}_3 - \mathbf{x}_3\mathbf{y}_2) + (\mathbf{x}_3\mathbf{y}_4 - \mathbf{x}_4\mathbf{y}_3) + (\mathbf{x}_4\mathbf{y}_1 - \mathbf{x}_1\mathbf{y}_4)]$$

Note : *The student is advised to avoid the direct application of the above formula in solving numerical questions. In each such case, he should divide the given quadrilateral into two triangles by drawing a diagonal, and then find the area by adding together the areas of the two triangles so formed.*

Solved Examples

Ex. 1. Find the area of the triangle the co-ordinates of whose vertices are (1, 3), (−7, 6), and (5, −1).

Sol. Here $x_1 = 1$, $x_2 = -7$, $x_3 = 5$

and $y_1 = 3$, $y_2 = 6$, $y_3 = -1$

∴ the required area

$$\begin{aligned}&= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\\ &= \tfrac{1}{2}[1(6+1) - 7(-1-3) + 5(3-6)]\\ &= \tfrac{1}{2}(7 + 28 - 15) = \tfrac{1}{2} \times 20 = 10.\end{aligned}$$

Ex. 2. Show that the points (−5, 1), (5, 5) and (10, 7) lie in a straight line.

Sol. Here $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$

$$\begin{aligned}&= -5(5-7) + 5(7-1) + 10(1-5)\\ &= 10 + 30 - 40 = 0.\end{aligned}$$

Hence the three points are collinear.

Ex. 3. If (x, y) be any point in the straight line which passes through (2, 4) and (5, 9), prove that

$$5x-3y+2=0.$$

Sol. This clearly means that the three points (x, y), (2, 4), and (5, 9) are collinear.

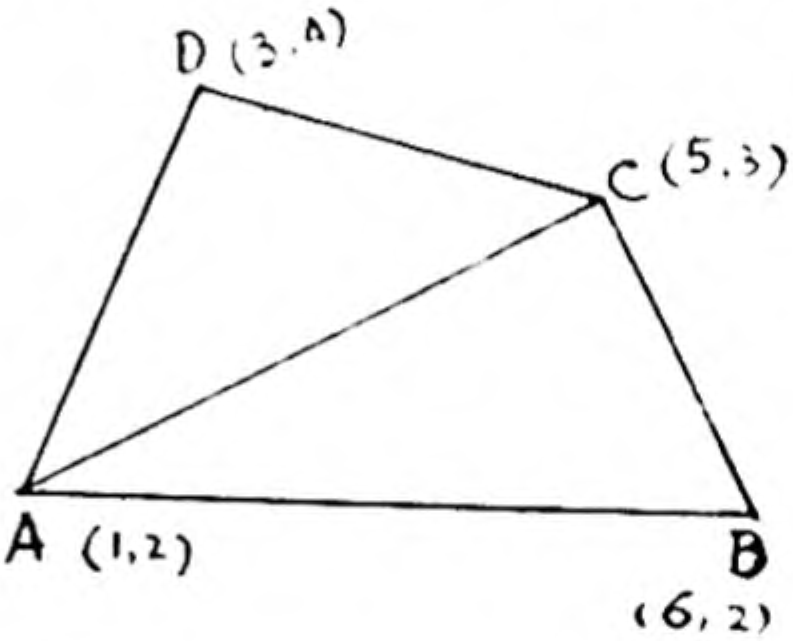
$$\therefore \quad x(4-9)+2(9-y)+5(y-4)=0$$

This gives $\quad 5x-3y+2=0.$

Ex. 4. Find the area of the quadrilateral whose angular points are (1, 2), (6, 2), (5, 3), and (3, 4).

Sol. Let ABCD be the diagonal. Join AC. Then the area of the quadrilateral

$$\begin{aligned}
&=\triangle ABC+\triangle ACD\\
&=\tfrac{1}{2}[1(2-3)+6(3-2)\\
&\quad+5(2-2)]\\
&\quad+\tfrac{1}{2}[1(3-4)+5(4-2)\\
&\quad+3(2-3)]\\
&=\tfrac{1}{2}[-1+6+0]\\
&\quad+\tfrac{1}{2}[-1+10-3]\\
&=\tfrac{5}{2}+3=\tfrac{11}{2}
\end{aligned}$$



Exercise 3

1. Find the area of the triangle whose vertices are :—
   - (*i*) (2, 7), (8, 4), (5, 10).
   - (*ii*) $(a \cos \alpha, b \sin \alpha)$, $(a \cos \beta, b \sin \beta)$, $(a \cos \gamma, b \cos \gamma)$.
   - (*iii*) (−2, 5), (7, 8), (5, −2).
   - (*iv*) (−2, 3), (−7, 5), (3, −5).
   - (*v*) $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$.
   - (*vi*) $\{at_1t_2, a(t_1+t_2)\}$, $\{at_2t_3, a(t_2+t_3)\}$, $\{at_3t_1, a(t_3+t_1)\}$.
   - (*vii*) $\left(am_1, \dfrac{a}{m_1}\right)$, $\left(am_2, \dfrac{a}{m_2}\right)$, $\left(am_3, \dfrac{a}{m_3}\right)$.
2. Show that the following sets of points are collinear :—
   - (*i*) (0, −7), (1, −5), (4, 1).
   - (*ii*) $(a, b+c)$, $(b, c+a)$, $(c, a+b)$.

(*iii*) (x_1, y_1), (x_2, y_2), $\left(\dfrac{lx_2+mx_1}{l+m}, \dfrac{ly_2+my_1}{l+m}\right)$.

(*iv*) (2, 3), (6, 9), (0, 0).

(*v*) $(a, 0)$, $(0, b)$, $(3a, -2b)$.

3. If the point (x, y) be collinear with the points $(1, -2)$ and $(3, -4)$, show that

$$x+y+1=0.$$

4. If (x, y) be any point in the straight line which passes through $(a, 0)$ and $(0, b)$ prove that

$$\frac{x}{a}+\frac{y}{b}=1.$$

5. Determine the value of k in order that the points $(2, -1)$, $(-3, 4)$, and $(k, 5)$ may be collinear.

6. A, B, C are the points $(-1, 5)$, $(3, 1)$, and $(5, 7)$ respectively. D, E, F are the mid-points of BC, CA, and AB respectively. Show that the area of the triangle ABC is four times that of the triangle DEF.

7. The co-ordinates of A, B, C are $(6, 3)$, $(-3, 5)$, $(4, -2)$ respectively, and P is any point (x, y), show that

$$\frac{\triangle \mathrm{PBC}}{\triangle \mathrm{ABC}}=\frac{x+y-2}{2}.$$

8. Find the area of the quadrilateral whose angular points are :—

(*i*) $(-5, 7)$, $(3, -2)$, $(2, -4)$, and $(-8, -9)$.

(*ii*) (1, 2), (6, 2), (5, 3), and (3, 4).

CHAPTER II

LOCUS

2.1. Def. *The* **locus** *of a moving point is defined as the path traced out by it in* **accordance** *with some given geometrical condition.*

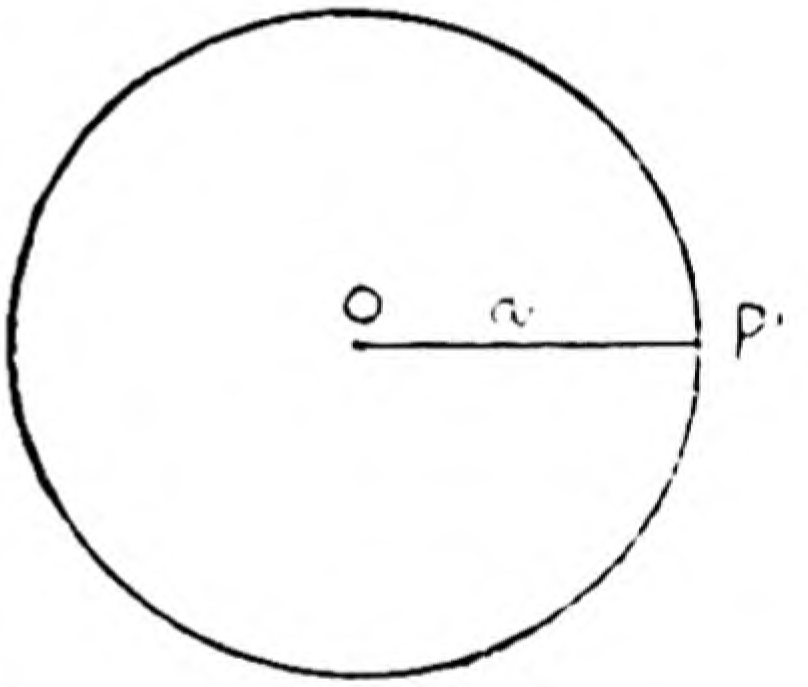
Mathematically, the locus is defined by the following equation :—

Locus=path traced out by the point+some given geometrical condition.

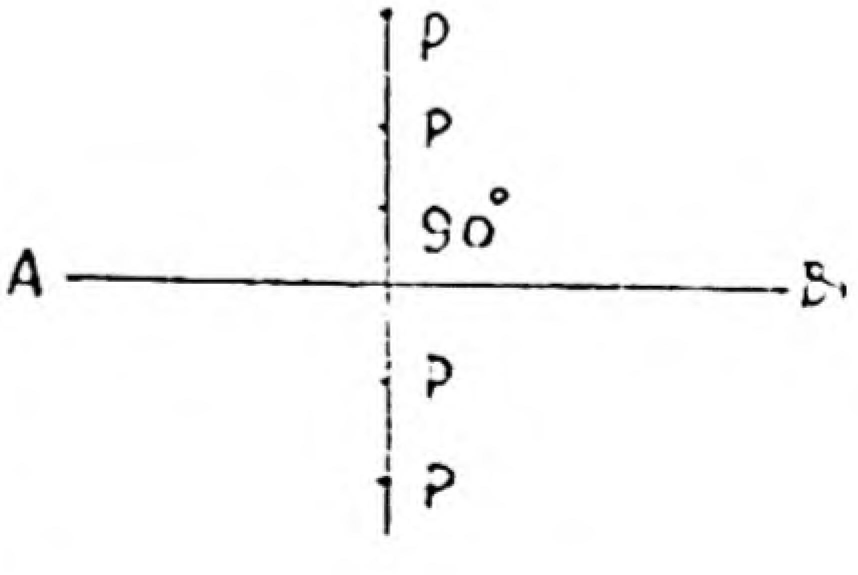
We can, therefore, say that *every* "locus" is a path, whereas *every* path may or may not be a locus.

Illustrations : —

(1) Let a point P move in such a way that it remains at a constant distance from a fixed point O. In this case, we can say that the locus (*i.e.*, the path traced out by the point P) is *the circumference of a circle* whose *centre* is O. If the constant distance be "a" units in length, we can say that the *radius* of this circle is "a".

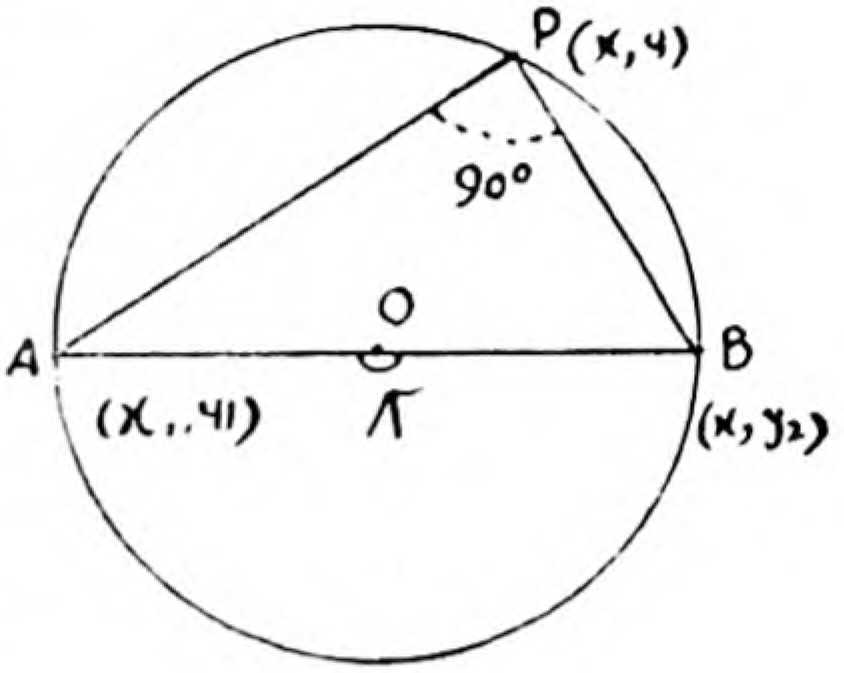


(2) Again, if there are two fixed point A and B in the plane of the paper, and if P be a moving point which moves in such *a way that it remains at equal distance* from these two fixed point, we can say that the locus of P is a straight line bisecting AB perpendicularly. In



other words, *the locus of P is the right bisector of the line AB.* P must lie on the right bisector of AB, as no other position of it will *satisfy* the given condition.

(*iii*) Let us take the case of a moving point P which moves so as to describe ∠APB = 90°, where A and B are two fixed points. If we describe a circle with AB as its diameter, then P will be on the circumference of the circle so described. In other words, the *locus* of P is a *circle* whose diameter is AB, as a angle in a semi-circle is always a right angle.



From the above illustrations it is obvious that the locus of a moving point is a curve which includes even a straight line.*

2.2. Def. "*The equation of a locus is defined as an equation in x and y and the constants, if any, which is satisfied by the co-ordinates of any and every point on the locus and by those of no other point.*"

To every geometrical locus, there will correspond an algebraic relation. Conversely, for any given algebraic relation in x, y there exists a geometrical locus such that every point whose co-ordinates satisfy the given algebraic relation *lies* on the *locus* and such that no other point whose co-ordinates do no *satisfy* the given algebraic relation lies on the locus. *This locus is called the curve represented by the given algebraic relation.*

The co-ordinates of a point moving along a curve are called its *current co-ordinates* and are *generally* denoted *by* (x, y). Co-ordinates of a point occupying a particular position are, on the other hand, denoted *by* (x_1, y_1) or (x', y').

---

* a straight line is called a st. curve.

2.3. How to find a locus ?

The locus of a point is to be found out as under : —

(*i*) *From the given data, obtain an algebraic relation in* x_1, y_1 *and the constants.*

(*ii*) *Replace* x_1, y_1 *by* x, y *respectively in the said algebraic relation.*

(*iii*) *The relation thus obtained will be the required locus.*

Note : If a point lies on the locus, its co-ordinates *must* satisfy the equation of the locus.

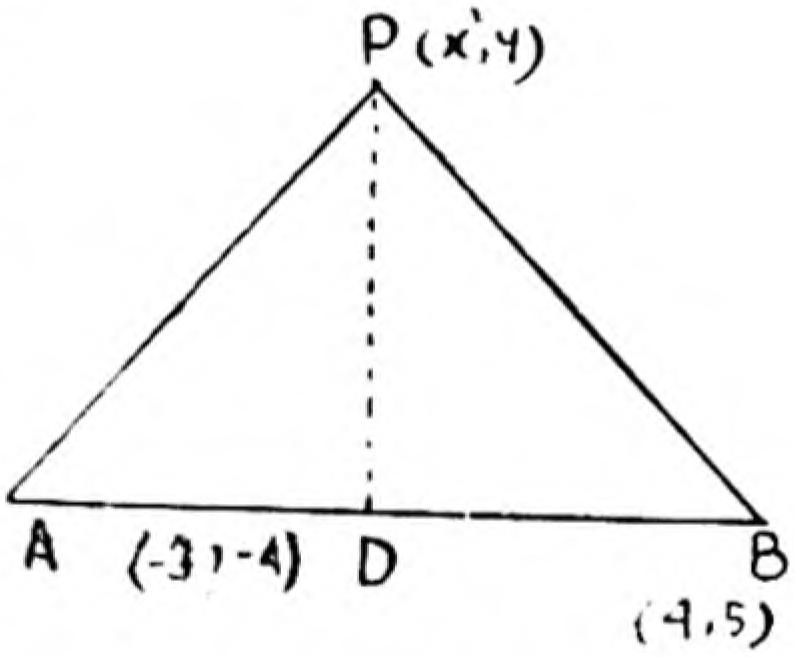
Solved Examples

Ex. 1. Find the locus of a point which moves such that it is always equidistant from the points (—3, —4) and (4, 5)

Sol. Let P $(x_1 y_1)$ be any particular position of the moving point. Now under the given geometrical condition, we have

$$PA=PB \qquad \ldots(i)$$

where the co-ordinates of A and B are (–3, –4) and (4, 5) respectively.



Now (*i*) gives

$$\sqrt{(x_1+3)^2+(y_1+4)^2}=\sqrt{(x_1-4)^2+(y_1-5)^2}$$

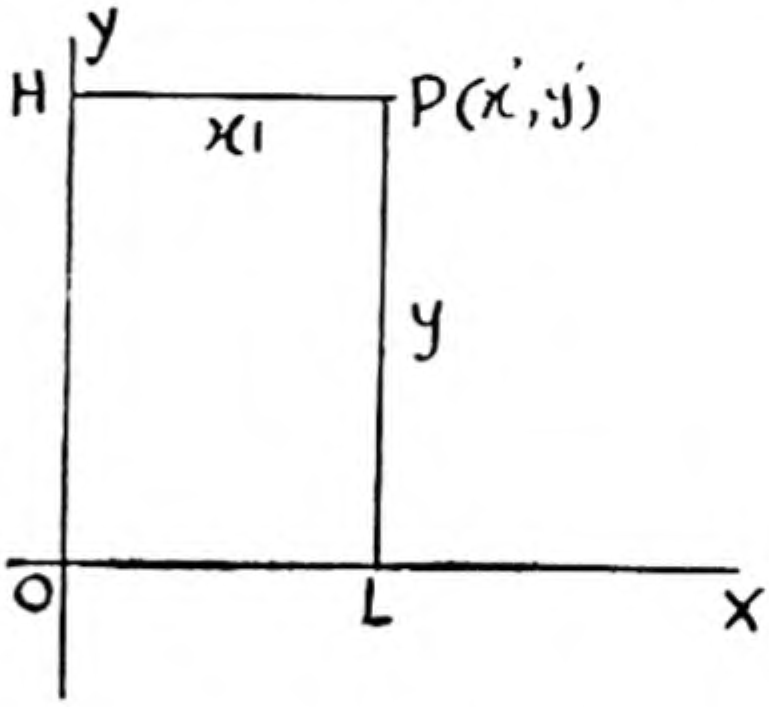
or $$7x_1+9y_1-8=0$$

This gives the algebraic relation in x_1, y_1. Hence, replacing x_1 by x and y_1 by y, we get the required locus as

$$7x+9y-8=0$$

Ex. 2. A point moves so that its distance from x-axis exceeds twice its distance from y-axis by 3. Find the equation of its locus.

Sol. Let P(x', y') be any particular position of the point P which lies on the locus. Now, the distance of P from x-axis $= y'$ and its distance from y-axis $= x'$



Under the given condition

$$y' = 2x' + 3$$

Which is the relation in x', y'. Hence the required locus is

$$y = 2x + 3$$

(Changing x' into x and y' into y)

Ex. 3. A point moves such that the sum of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$. Prove that its locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{when } b^2 = a^2(1 - e^2)$$

Sol. Let P(x_1, y_1) be any particular position of the moving point, and the co-ordinates of two fixed points A and B be $(ae, 0)$ and $(-ae, 0)$ respectively.

Then under the given geometrical condition we have

$$\sqrt{(x_1 - ae)^2 + y_1^2} + \sqrt{(x_1 + ae) + y_1^2} = 2a \qquad \ldots(i)$$

Now *identically*

$$[(x_1 - ae)^2 + y_1^2] - [(x_1 + ae)^2 + y_1^2] = -4aex_1 \qquad \ldots(ii)$$

Dividing (2) by (1) we get

$$\sqrt{(x_1 - ae)^2 + y_1^2} - \sqrt{(x_1 + ae)^2 + y_1^2} = -2ex_1 \qquad \ldots(iii)$$

Adding (1) and (3) we have

$$2\sqrt{(x_1 - ae)^2 + y_1^2} = 2a - 2ex_1$$

or $\sqrt{(x_1 - ae)^2 + y_1^2} = a - ex_1$

Squaring this, we get

$$(x_1 - ae)^2 + y_1^2 = a^2 + e^2x_1^2 - 2aex_1$$

or $x_1^2(1 - e^2) + y_1^2 = a^2(1 - e^2)$

or $$\frac{x_1^2}{a^2}+\frac{y_1^2}{a^2(1-e^2)}=1 \qquad \text{[dividing by } a^2(1-e^2)\text{]}$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $a^2(1-e^2)=b^2$

which gives the required locus.

2.4. From the examples worked out above, it is clear that the locus of a moving point is an equation in x and y. Such an equation need not have both x and y : it may have only x or only y or both. Even the degrees of the equations vary; these are different for different loci*. For instance, a first degree equation in x, and y represents a straight line, whereas a second degree equation may represent a circle if it satisfies certain conditions, and so on. Thus the equation of a straight line or that of a circle will be a locus, no matter whether it represents a straight line or a circle. The same is the case with other curves such as Parabola, Ellipse, or a Hyperbola.

Thus we conclude that every geometrical curve has an equation and every algebraic equation represents a curve.

Exercise 4

1. Find the locus of a point which moves such that its distance from the point (1, 2) is equal to its distance from the axis of y.

2. Find the equation to the locus of a point which is always equidistant from the points whose co-ordinates are :—

   (*i*) (1, 0) and (0, —2).

   (*ii*) (2, 3) and (4, 5).

   (*iii*) $(a+b, a-b)$ and $(a-b, a+b)$.

3. Find the locus of a point which moves so that its distance from the axis of x is three times its distance from the axis of y.

4. A point P moves in such a way that its distance from the point $(a, 0)$ is always four times its distance from the axis of y.

---

*Loci is the plural of Locus.

5. Find the locus of a point the sum of the squares of whose distances from the axes is equal to 3.

6. A and B being the fixed points $(a, 0)$ and $(-a, 0)$ respectively, obtain the equations giving the locus of P, when :—

(*i*) $PA^2 - PB^2 = 2k^2$ (a constant).

(*ii*) $PA + PB = c$ (a constant).

(*iii*) $PA = n.PB$ when n is a constant.

(*iv*) $PB^2 + PC^2 = 2PA^2$ where C is a point whose co-ordinates are $(c, 0)$.

7. Show that the locus of a point P which moves in such a way that its distance from the axis of x is always one-half its distance from the origin is

$$x^2 = 3y^2.$$

8. A point moves so that it is always collinear with the points $(2, 3)$ and $(-4, 5)$; find its locus.

9. A straight line of length $2d$ has its ends on the co-ordinate axes. Find the equation of the locus of its middle point.

[**Hint :** Let A and B be the ends of the st. line and H its mid-point. Then $OM = AM = MB = d$ etc.]

Imp.

10. A point moves such that the difference of its distances from the points $(c, 0)$ and $(-c, 0)$ is $2a$. Show that its locus is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

11. P and Q are two variable points on the axis of x and y respectively, such that $OP + OQ = a$; find the locus of the foot of the perpendicular from the origin on PQ.

12. A straight line AB, of fixed length, slides between two perpendicular lines OX and OY in such a way that the point A always lies on OX and the point B on OY. Find the locus of the point P which divides AB into two parts PA and PB such that $PA = a$ and $PB = b$.

CHAPTER III

THE STRAIGHT LINE

3.1. In Article 2.4 of the preceding chapter, we have discussed the significance of the locus of a moving point. It has been pointed out there that the locus of a point may represent a straight line, a circle, or some other geometrical curve. Here in the present chapter we are now concerned with the equation of a straight line. However, before explaining various forms of a straight line, we propose to show in the following article that an equation of the first degree in x and y *always* represents a *straight line.*

3.2. *To prove that an equation of the first degree in x and y represents a straight line.*

The *most general* form of such an equation is

$$ax+by+c=0 \qquad \ldots(1)$$

where a, b, c are constants, *i.e.*, quantities which do not contain x or y and which remain the same for all points on the locus.

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be *any* three points on the *locus* represented by equation (1).

Since the point (x_1, y_1) lies on the locus, its co-ordinates when substituted for x and y in (1) must ratisfy it.

$\therefore$ $\qquad ax_1+by_1+c=0 \qquad \ldots(2)$

Similarly, $\qquad ax_2+by_2+c=0 \qquad \ldots(3)$

and $\qquad ax_3+by_3+c=0 \qquad \ldots(4)$

Using the determinant notation to eliminate a, b, c from (2), (3), and (4), we have :—

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad ..(5)$$

Now (5) when expanded gives

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0 \qquad ...(6)$$

But in Article 1.16, we have seen that (6) is condition under which the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) can be *collinear* (*i.e.*, lie in a straight line). **The locus must, therefore, represent a straight line.**

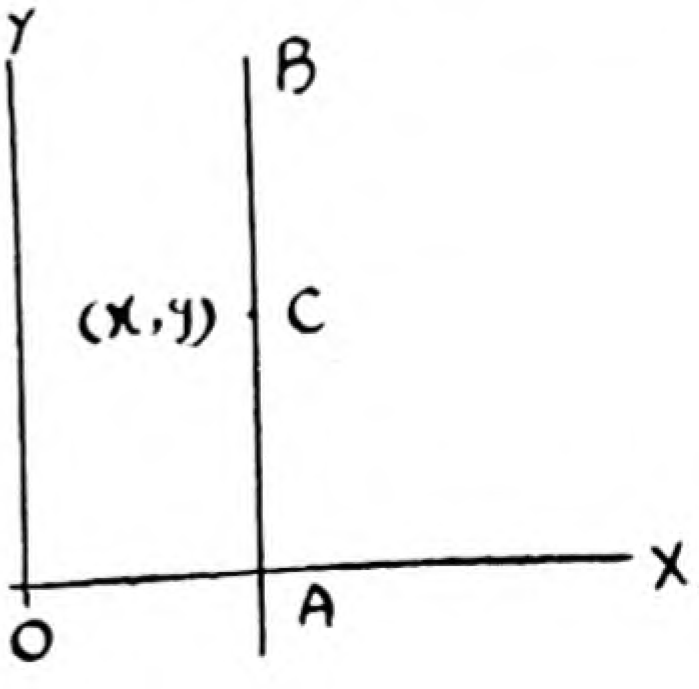
Note : (1) A first degree equation *need not* contain both x and y ; it *does* represent a straight line even if it contains only one of them, as will be seen later on.

(2) An alternative proof for the above article will be given in Article 3.8.

3.3. *To find the equation to a straight line parallel to one of the co-ordinate axes.*

(*a*) *Equation to a straight line parallel to the axis of* y.

Let AB be any straight line parallel to the axis of y cutting OX in A, so that OA$=a$. Let C be *any* other point on AB with (x, y) as its co-ordinates.



Then the abscissa of C is always "a" so that

$$\mathbf{x=a} \qquad ...(1)$$

This is true for every point lying on the line AB, and is not true for *any* other point. *Hence (1) represents* *the equation to the line which is parallel to y-axis at a distance "a" from it.*

Note : We can similarly show that

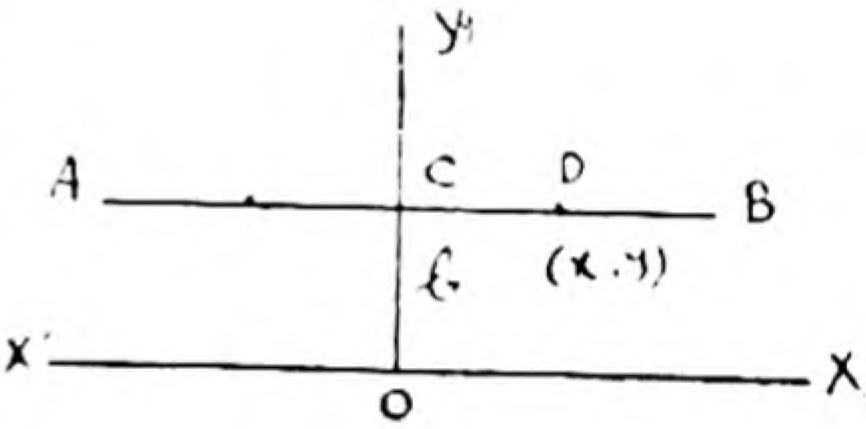
$$\mathbf{x + a = 0}$$

is the question to a straight line which is parallel to the axis of y at a distance "$-a$" from it. In this case the line AB will lie to the left of the origin O.

(*b*) *Equation to a straight line parallel to the axis of x.*

Let AB be the given line parallel to the x-axis cutting OY in C, so that $OC = b$. Let D be *any* other point on AB with co-ordinates x and y. Then the ordinate of D is always "b", so that

$$\mathbf{y = b} \qquad \dots(2)$$



This relation is true for every point lying on AB and not for any other point. *Hence* (2) *represents the equation to the straight line which is parallel to the axis of x at a distance "b" from it.*

Note : We can likewise show that

$$\mathbf{y + b = 0}$$

is the equation to a straight line which is parallel to the axis of x at a distance "$-b$" from it. In this case, the line AB will be below the origin O.

Cor. *The equations to the x-axis and the y-axis are $y = 0$ and $x = 0$ respectively.*

3.4. Slope of a line

Def. *The slope (or gradient) of a line is the tangent of the angle which the part of the line above the x-axis makes with the positive direction of the x-axis.*

Explanation :

In figures (*i*) and (*ii*) AB is the part of the line above the x-axis and OX is the positive direction of the x-axis. Let $\angle XAB=\theta$, **then the slope of the line** $=$ **tan** θ

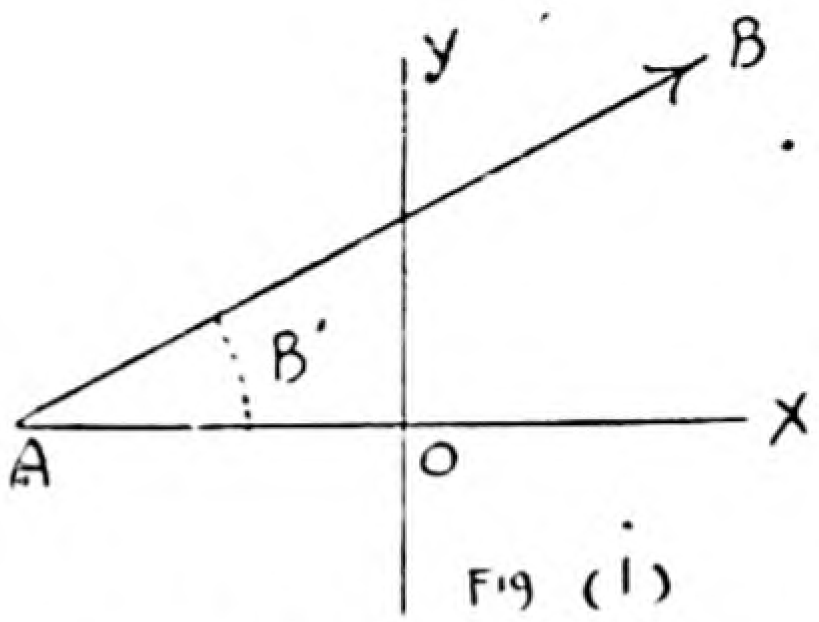


Fig (i)

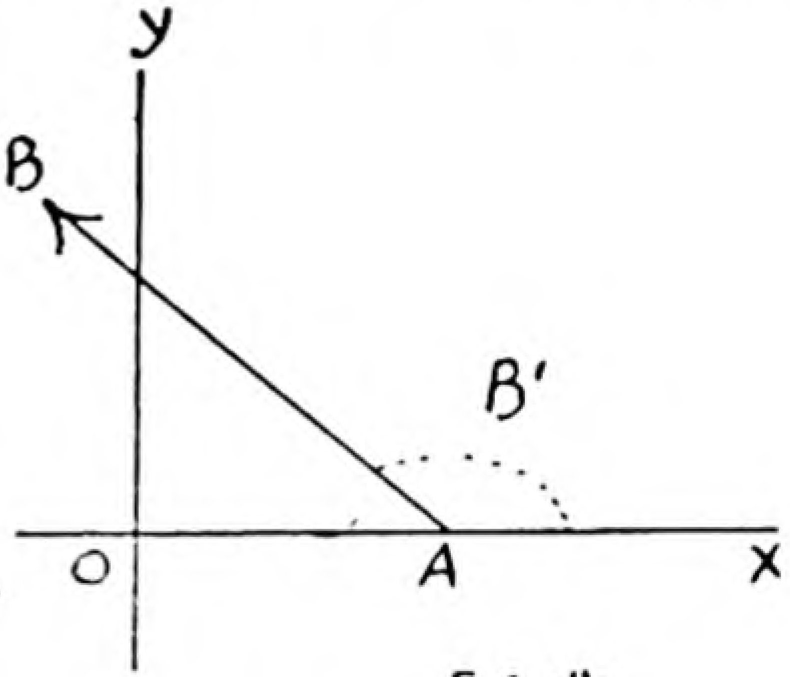


Fig (ii)

Notation :—

The slope of a straight line is **generally** *denoted by "m"*

Thus

$$\tan\theta=m$$

Note : The student is advised to understand thoroughly as to what is meant by "the angle which the part of the line above the x-axis makes with the positive direction of the x-axis." This means the angle measured in the counter-clockwise direction from the position of the x-axis to the part of the line above the

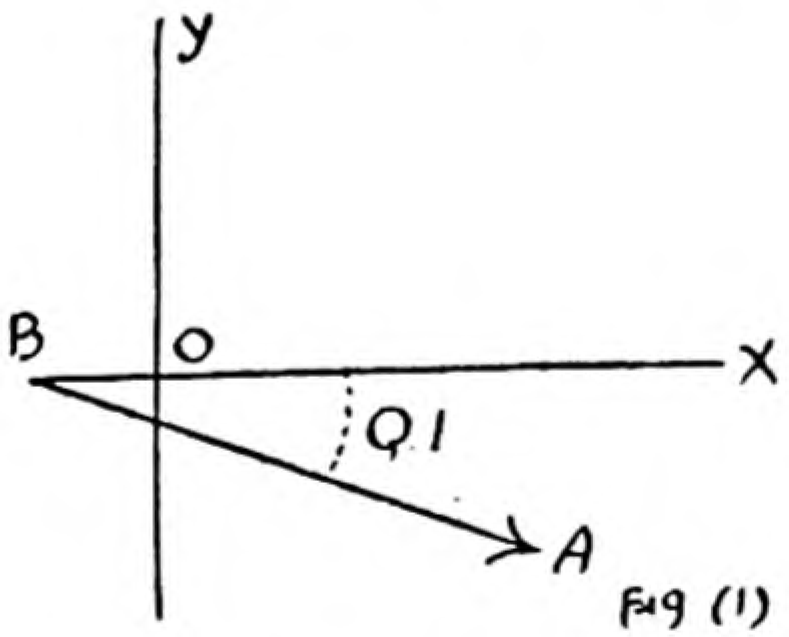


Fig (1)

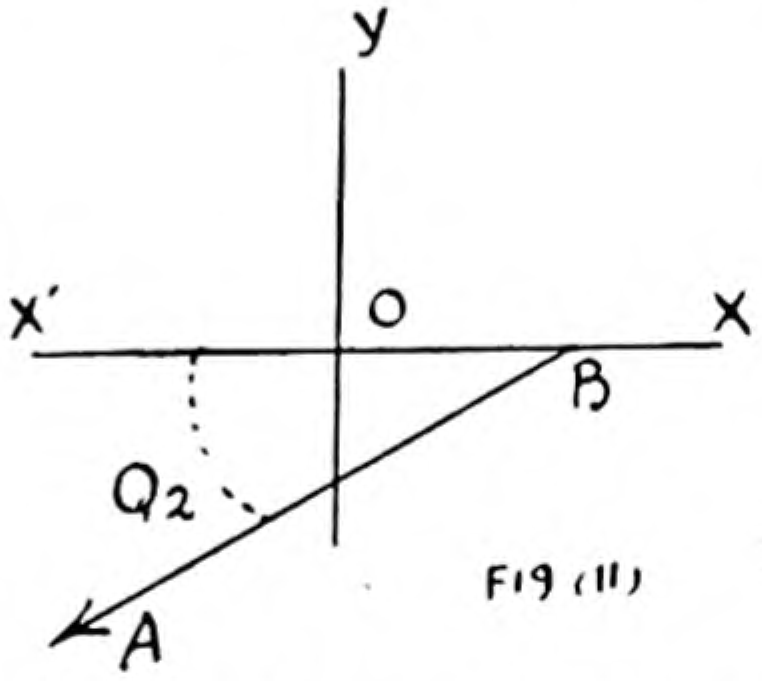
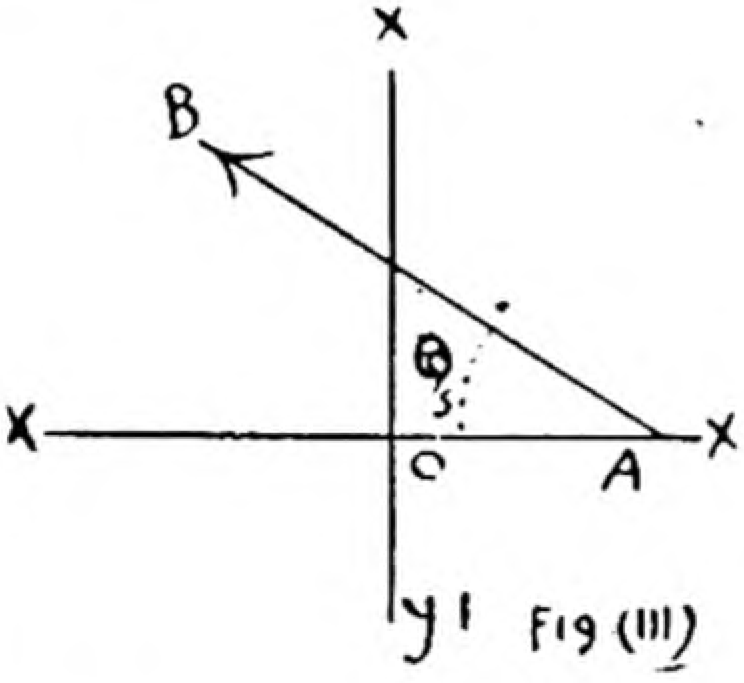


Fig (ii)

x-axis. Tangents of the angles θ_1, θ_2 and θ_3 of the following *cannot*, therefore, be called slopes.



In fig. (*i*), θ_1 though made with the positive direction OX of the x axis is measured in the *clockwise* direction, and tan θ_1 cannot be called the *slope* of AB as such.

In figure (*ii*), θ_2 is made with OX′, the −ve direction of the x-axis.

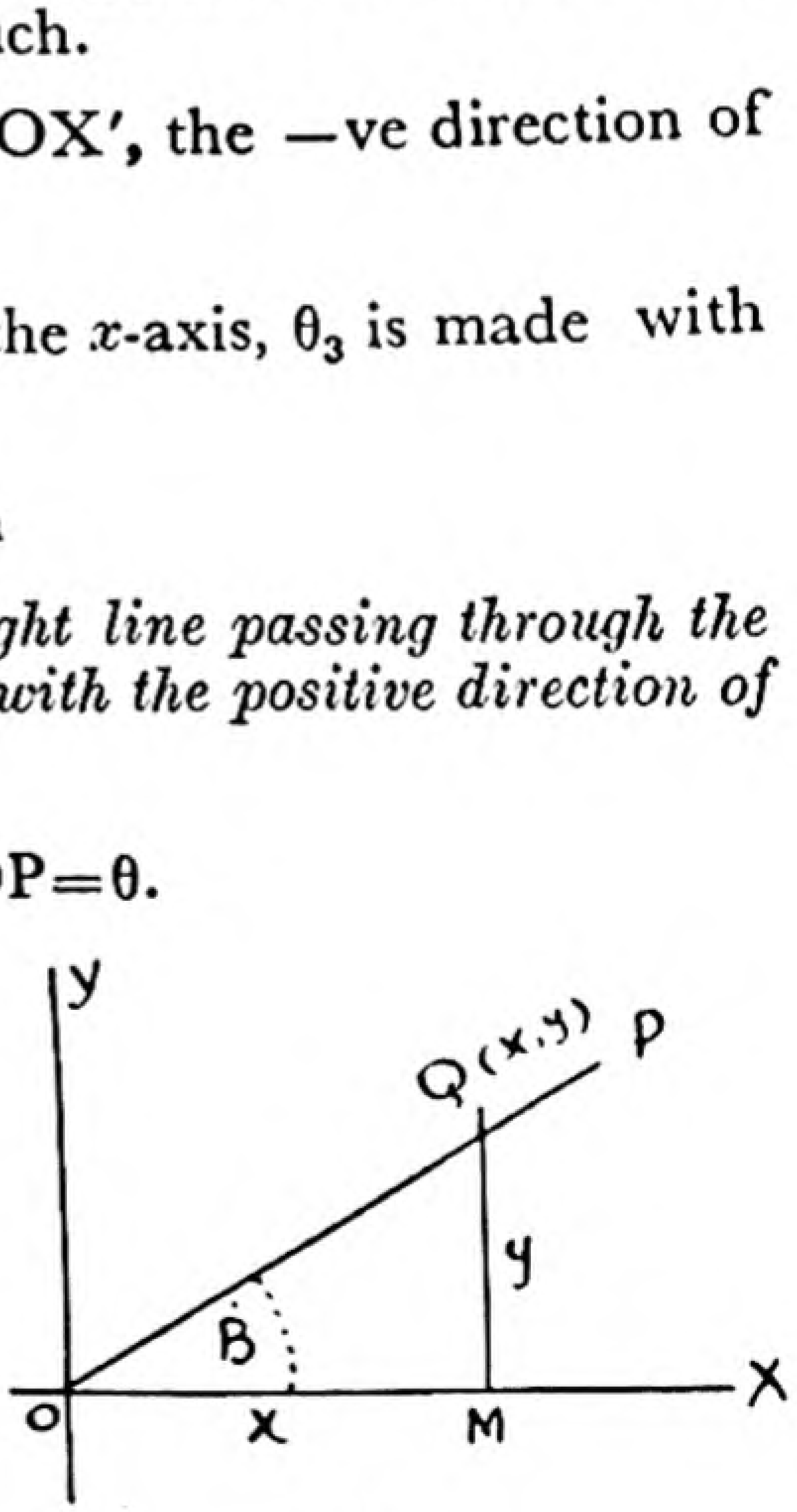
In fig. (*iii*), though AB is above the x-axis, θ_3 is made with the −ve direction of the x-axis.

3.5. A line through the origin

To find the equation of a straight line passing through the origin and making a given angle θ with the positive direction of the x-axis.

Let OP be the line, so that $\angle XOP = \theta$.

Take $Q(x, y)$ *any point* on OP, and QM perpendicular on OX.



$$\text{Now } \frac{QM}{OM} = \tan\theta$$

$$\text{or} \quad QM = OM \tan\theta \quad \ldots(1)$$

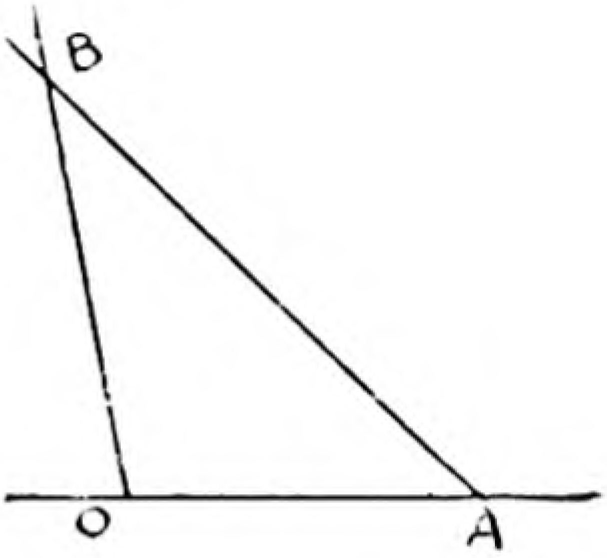
But $QM = y$ and $OM = x$

Let $\tan\theta = m$

∴ (1) gives

$\mathbf{y = mx}$, which is the required equation.

3.6. Intercepts on the axis



Let a straight line AB cut the x-axis in A and the y-axis in B. Then

(*i*) *OA is called the intercept of the straight line on the x-axis.* And

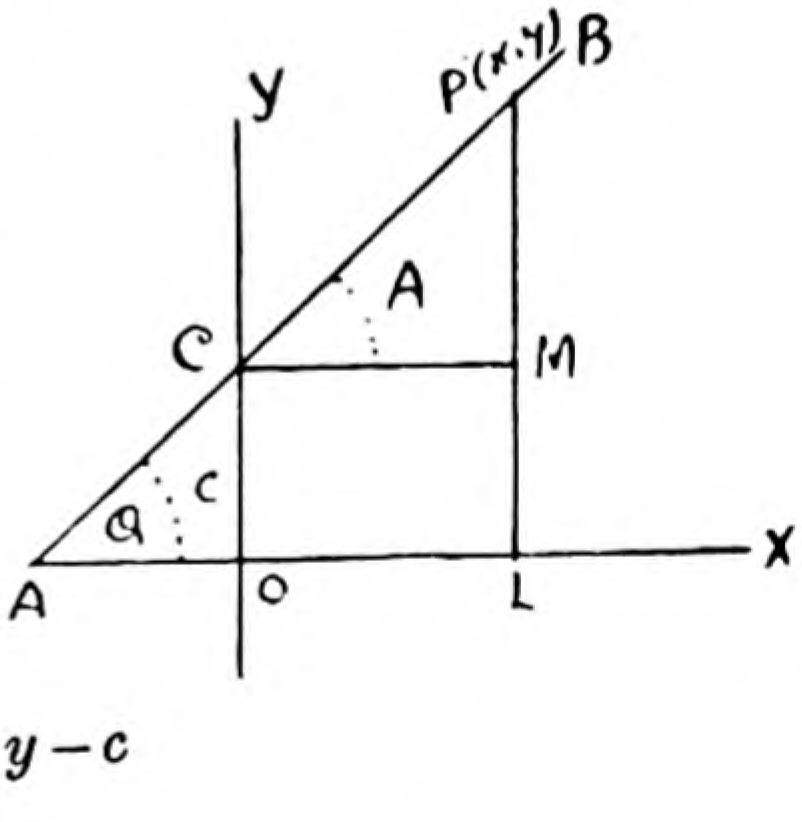
(*ii*) *OB is called the intercept of the straight line on the y-axis.*

The intercept on the x-axis is *positive* if measured to the right of the origin, and *negative* if measured to the left.

The intercept on the y-axis is *positive* if measured above the origin and *negative* if measured below.

3.7. Slope form of a straight line

To find the equation of a straight line which cuts off a given intercept "C" on the y-axis and is inclined at an angle θ to the positive direction of the x-axis.



Let AB be the line meeting the axis of x and y in A and C respectively, so that $OC = c$ and $\angle XAC = \theta$.

Take any point $P(x, y)$ on AB, and draw PL perpendicular on OX and CM perpendicular on PL.

Now $PM = PL - LM = PL - OC = y - c$

and $CM = OL = x$

Therefore, from the right-angled triangle CMP,

$$\tan\theta = \frac{PM}{OL} = \frac{y-c}{x}$$

which gives

$$x\tan\theta = y - c$$

or
$$y = \tan\theta . x + c$$

Putting $\tan\theta = m$, we have

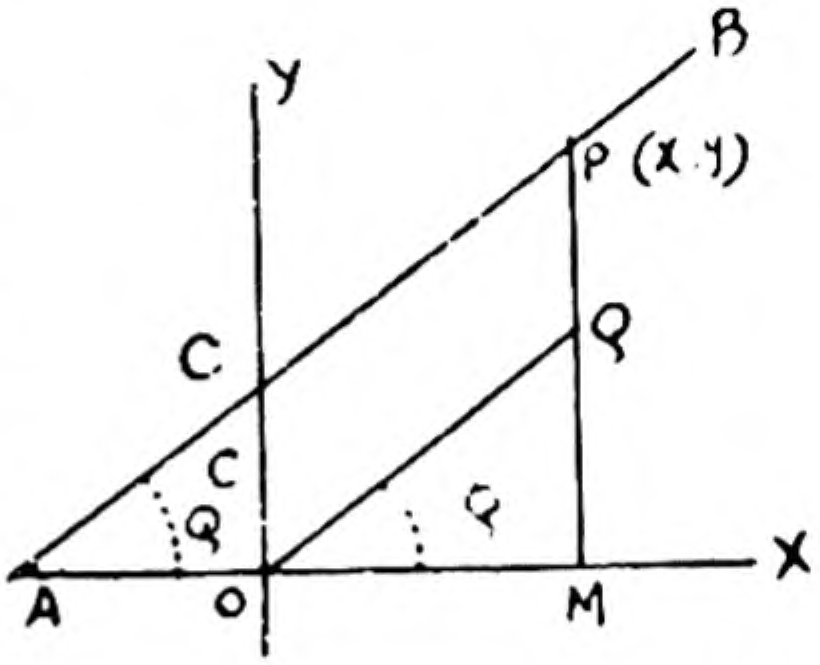
$$\mathbf{y = mx + c},$$ which gives the equation of the line.

Alternative Method

Let AB be the straight line meeting OY in C such that $\angle XOC = \theta$ and $OC = c$.

Take $P(x, y)$ any point on AB and draw PM perpendicular on OX. Through O, draw OQ parallel to AB meeting PM in Q. Then

$$\angle MOQ = \angle MAP = \theta \quad \text{(corresp.}\angle\text{s)}$$



Now $PM = PQ + QM = OC + QM$ ...(1)

But $$\frac{QM}{OM} = \tan\theta$$

or $$QM = OM\tan\theta$$

$\therefore$ (1) gives

$$PM = OC + OM\tan\theta$$

Putting $\tan\theta = m$, we get

$$\mathbf{y = c + mx} \qquad [\because PM = y.\ OM = x, \text{ and } OC = c]$$

or $$\mathbf{y = mx + c}$$

which gives the required equation of the straight line.

Note : The *slope form* of the equation of a straight line is also called the *tangent form* of the straight line.

3.8. To prove that a first degree equation in x and y represents a straight line. (Method of Comparison).

Proof. The most general equation of the first degree in x and y is

$$ax + by + c = 0 \qquad ...(1)$$

Case I. If $b \neq 0$, then (1) becomes

$$y = -\frac{a}{b}x - \frac{c}{b}$$

which being of the form

$$y = mx + c$$

represents a straight line.

Case II. If $b = 0$, then (1) becomes

$$ax + c = 0$$

or $$x = -\frac{c}{a}$$

which being of the form

$$x = k$$

represents a straight line parallel to the y-axis at a distance "$-\frac{c}{a}$" from it.

Hence $ax + by + c = 0$ *represents a st. line.*

Note : A more rigorous proof of the above theorem has been given in Article 3.2.

3.9. *To reduce the equation* $ax + by + c = 0$ *(general form) to the form* $y = mx + c$.

The equation $ax + by + c = 0$

Transposing, $by = -ax - c$

Dividing out by b, $y = -\frac{a}{b}x - \frac{c}{b}$

which gives the required form.

Note : (*i*) **Rule to reduce the general equation of a straight line to the slope form.**

Solve the given equation for y *(in terms of* x*).*

(*ii*) **Rule to write down the slope of a line whose equation is in the general form** *i.e*, **as** $ax + by + c = 0$

$$\textbf{Slope} = -\frac{\textbf{co-efficient of x}}{\textbf{co-efficient of y}}$$

Example :—What is the slope of the line whose equation is

$$3x+4y-5=0 \quad ?$$

$$\text{Slope}=-\frac{\text{coeff. of } x}{\text{coeff. of } y}=-\frac{3}{4}$$

Imp.

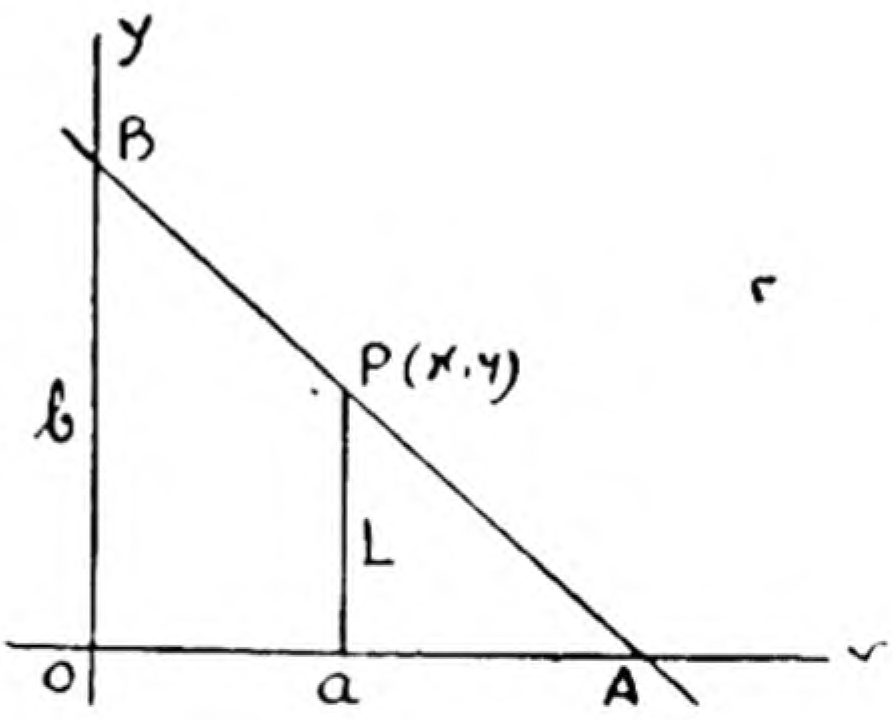
3.10. Intercept form of a straight line.

To find the equation of a straight line which cuts off given intercepts 'a' and 'b' from the axis.

Let AB be the given line meeting the axes in A and B, so that OA$=a$, OB$=b$.

Take P(x, y) any point on AB From P, draw PL perpendicular on OA.

Then from the similar △s, PLA and BOA



$$\frac{\text{LP}}{\text{OB}}=\frac{\text{LA}}{\text{OA}}=\frac{\text{OA}-\text{OL}}{\text{OA}} \qquad \ldots(1)$$

But LP$=y$, OB$=b$, OA$=a$, OL$=x$

∴ from (1), we get

$$\frac{y}{b}=\frac{a-x}{a}=1-\frac{x}{a}$$

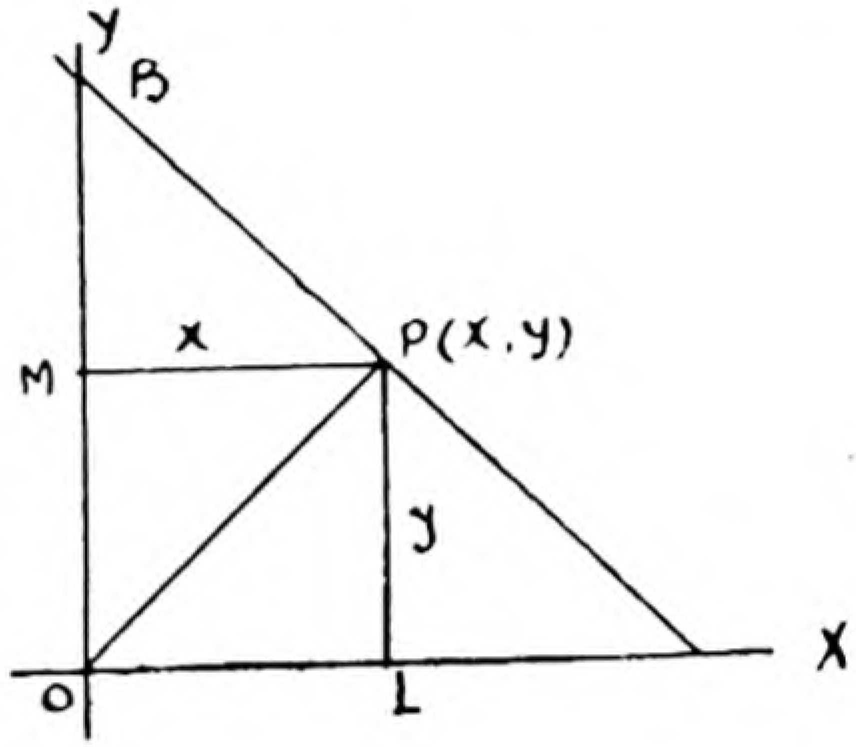
or

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=1$$

which is the required form.

Alternative Method :—(to be preferred)

Take P(x, y) any point on the line AB which meets the x-axis in A and the y-axis in B, so that OA$=a$ and OB$=b$.



Draw PL, PM ⊥s on OX and OY respectively. Join OP.

Now $\triangle BOA = \triangle BOP + \triangle POA$

or $\frac{1}{2}OB.OA = \frac{1}{2}OB.PM + \frac{1}{2}OA.PL$

or $\frac{1}{2}b.a = \frac{1}{2}b.x + \frac{1}{2}a.y$

or $ab = bx + ay$

Dividing out by ab, we have

$$1=\frac{x}{a}=\frac{y}{b}$$

or $$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}$$

which is required form.

Note : The line $\frac{x}{a}+\frac{y}{b}=1$ meets the x-axis in A (a, 0) and the y-axis in B(0, b).

3.11. *To reduce the equation* $ax+by+c=0$ *to the form* $\frac{x}{a}+\frac{y}{b}=1$ (Intercept form)

In the intercept form we notice that (i) the constant term is *unity* and the same occurs *on the R.H.S.*, and (ii) the coefficients of x and y occur as the *denominators* of x and y.

The equation is $ax+by+c=0$

(i) Transposing, $ax+by=-c$

(ii) Dividing out by $-c$, we get

$$\frac{a}{-c}x+\frac{b}{-c}y=1$$

(*iii*) then $\dfrac{x}{-\dfrac{c}{a}}+\dfrac{y}{-\dfrac{c}{b}}=1$ gives the required form.

Note : (1) Rule to reduce the general equation of a straight line to the intercept form.

(*i*) *Transpose the constant term to the R.H.S., and divide throughout by it.*

(*ii*) *Write the co-efficients of x and y as their denominators.*

(2) **Rule to find the intercepts of a line on the axes.**

(*i*) *In the equation $ax+by+c=0$, put $y=0$, so that $x=-\dfrac{c}{a}$.*

This will give the intercept made by the line on the x-axis.

(*ii*) *Put $x=0$ in the same equation, so that $y=-\dfrac{c}{b}$. This will give the intercept made by the line on the y-axis.*

(3) **Rule to trace the straight line by the equation $\mathbf{ax+by+c=0}$.**

(*a*) (*i*) *Put the given equation in the form*

$$\frac{x}{-\dfrac{c}{a}}+\frac{y}{-\dfrac{c}{b}}=1$$

(*ii*) *From OX', cut off $OA=-\dfrac{c}{a}$, and OY' cut off $OB=-\dfrac{c}{b}$*

(*iii*) *Join AB*

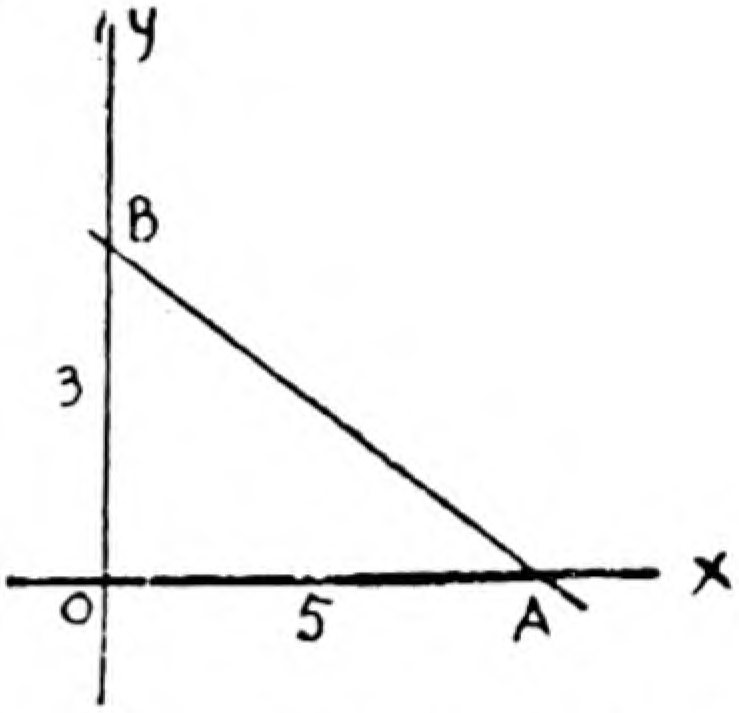
Then AB will represent the given line.

Example : Trace the line whose equation is
$3x+5y-15=0$.

Sol. The given equation can be put as

$$3x+5y=15$$

or $$\frac{x}{5}+\frac{y}{3}=1$$



From OX, cut off OA=5 and from OY, cut off OB=3. Join AB. Then AB will represent the given line.

(*b*) **The line $ax+by+c=0$ can be traced even as follows :—**

(i) Put $y=0$, so that $x=-\frac{c}{a}$

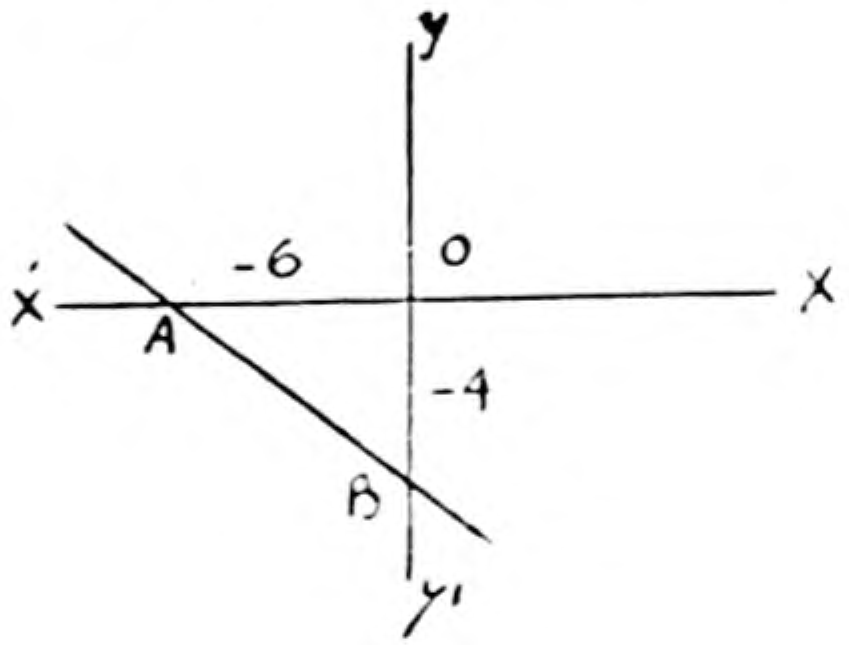
(ii) Put $x=0$, so that $y=-\frac{c}{b}$

(iii) From OX′, cut off $OA=-\frac{c}{a}$, and from OY′. cut off $OB=-\frac{c}{b}$

(iv) Join AB. Then AB will represent the given line.

Example : Trace the line
$2x+3y+12=0$.

Sol. Putting $y=0$, we get $x=-6$, and putting $x=0$, we get $y=-4$ from OX′, cut off OA=−6 and from OY′ cut off OB=−4.



Join AB which will *represent* the given line.

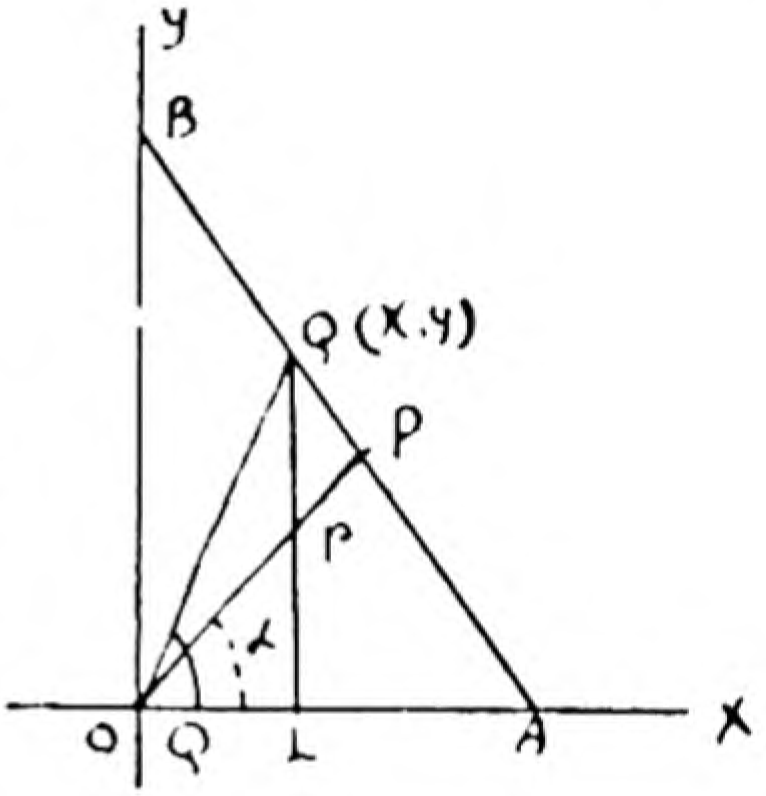
Note : *The student is advised to understand the rule given in (b) above as thoroughly as possible. This is preferable to the one given in (a).*

Impt

3·12. Perpendicular form of a straight line.

To find the equation of a straight line in terms of the perpendicular let fall upon it from the origin, and the angle that this perpendicular makes with the axis of x.

First Method :



Let OP$=p$ be the perpendicular from O upon AB, so that $\angle$AOP$=\alpha$.

Take Q(x, y) *any* point on AB and draw QL $\perp$ on OX. Let $\angle$LOQ$=\theta$, so that $\angle$POQ $=\theta-\alpha$.

Now from the right angled triangle POQ (rt.$\angle d$ at P), we have,

$$\frac{OP}{OQ}=\cos(\theta-\alpha)$$

or $$OP=OQ\cos(\theta-\alpha)$$

or $$OP=OQ\cos\theta\cos\alpha+OQ\sin\theta\sin\alpha \qquad \ldots(i)$$

Again, from right-$\angle d$, $\triangle$LOQ, we get

$$\frac{QL}{OQ}=\sin\theta$$

or $$QL=OQ\sin\theta$$

or $$y=OQ\sin\theta \qquad (\because QL=y)$$

also $$\frac{OL}{OQ}=\cos\theta$$

or $$OL=OQ\cos\theta$$

or $$x=OQ\cos\theta \qquad (\because OL=x)$$

Substituting for OQ cos θ and OQ sin θ in (i) and *remembering* that OP$=p$, we obtain

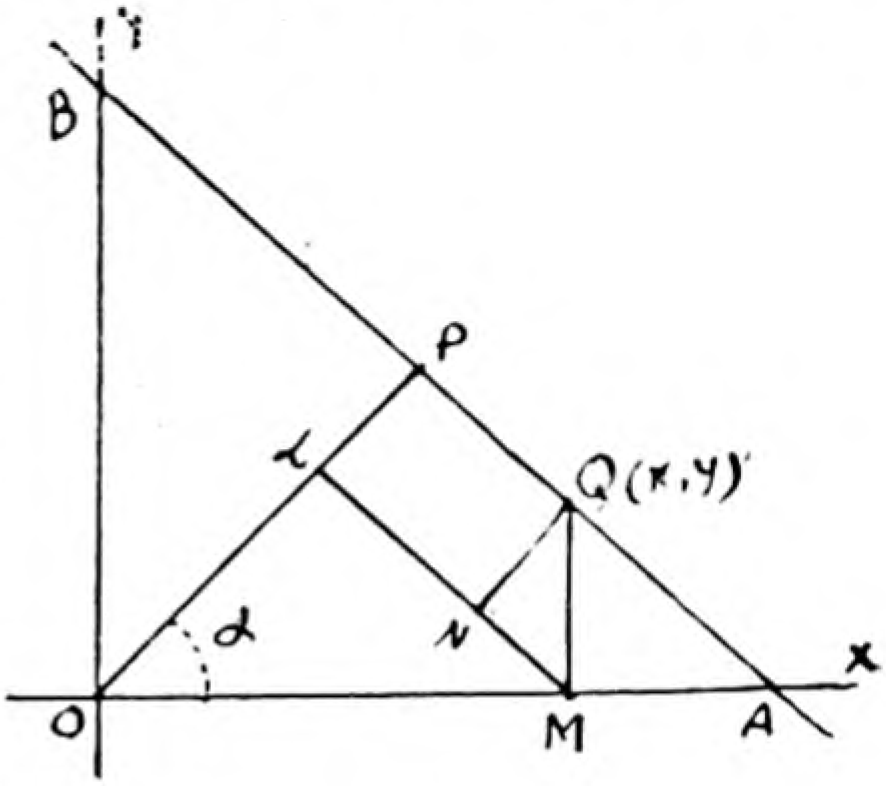
$$p=x\cos\alpha+y\sin\alpha$$

or $\mathbf{x \cos \alpha + y \sin \alpha = p}$

which gives the required form.

Second Method : (*To be preferred*)

Let $OP = p$ be the perpendicular from O upon AB, and α be the angle which OP makes with the x-axis.



Take $Q(x, y)$ any point on AB.

Draw $QM \perp OX$ and $QN \perp$ on LM

Now $OL = OM \cos \alpha$ ...(1)

$LP = QN$

$= QM \sin QMN$

But $\angle QMN = 90° - \angle LMO$

$= \angle LOM = \alpha$

$\therefore$ $LP = QM \sin \alpha$ ...(2)

Adding (1) and (2) we get

$$OM \cos \alpha + QM \sin \alpha = OL + LP = OP$$

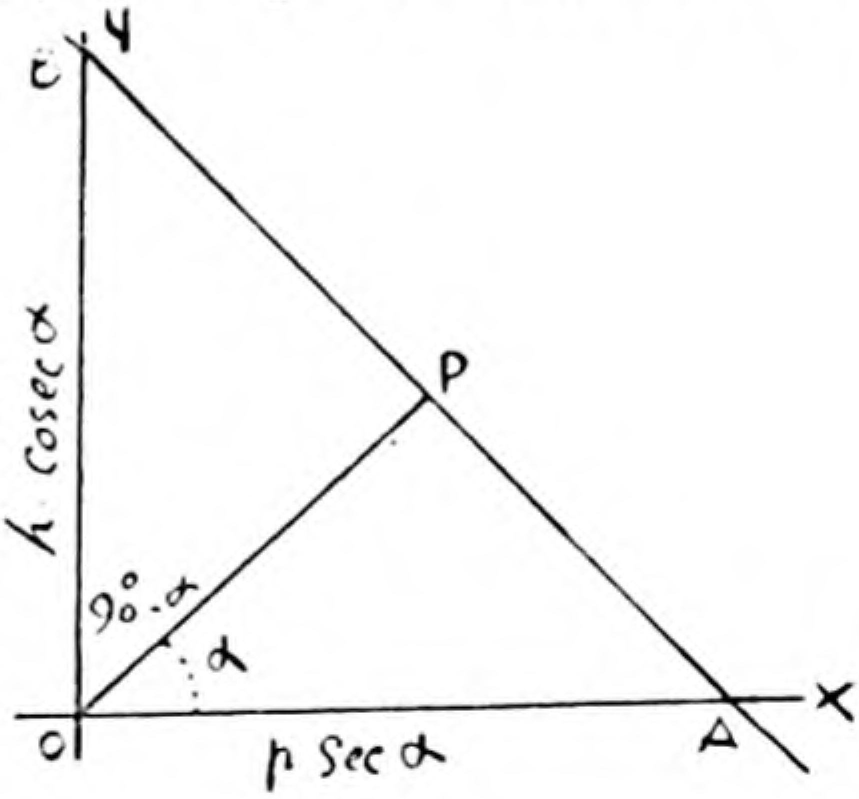
i.e., $\mathbf{x \cos \alpha + y \sin \alpha = p}$ $[\because OM = x, QM = y, OP = p]$

which is the required form.

Third Method :

From the right angled $\triangle$ AOP, $OA = p \sec \alpha$ and from the right-angled $\triangle$ POB,

$$OB = p \operatorname{cosec} \alpha$$



Now $p \sec \alpha$ and $p \operatorname{cosec} \alpha$ are the intercepts which the line AB makes on the axis of x and y respectively.

Therefore, the equation of AB in the intercept form will be as follows :

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1 \qquad \text{(Article 3.10)}$$

or $$\frac{x}{\sec\alpha}+\frac{y}{\operatorname{cosec}\alpha}=p$$

or $$\mathbf{x\cos\alpha+y\sin\alpha=p.}$$

Note : *The above form of the straight line is called the Normal form as well.*

3.13. To reduce the equation ax+by+c=0 to the form x cos α+y sin α=p (Perpendicular form).

In the form $x\cos\alpha+y\sin\alpha=p$, we notice that the sum of the squares of the co-efficients of x and y is *unity* and that the constant term on the R.H.S. *i.e.*, "p" is *positive*. Therefore, we will multiply the equation $ax+by+c=0$ throughout by a suitable factor, so that the

(co-efficient of x)²+(co-efficient of y)²=1.

Let h be that factor. Therefore, on multiplying

$$ax+by+c=0$$

throughout by h, we have

$$ahx+bhy+ch=0 \qquad \ldots(1)$$

Now $$(ah)^2+(bh)^2=1$$

This gives $$h=\frac{1}{\sqrt{a^2+b^2}}$$

Thus (1) becomes (after substituting for h)

$$\frac{a}{\sqrt{a^2+b^2}}\,x+\frac{b}{\sqrt{a^2+b^2}}\,y+\frac{c}{\sqrt{a^2+b^2}}=0$$

or $$\frac{a}{\sqrt{a^2+b^2}}\,x+\frac{b}{\sqrt{a^2+b^2}}\,y=-\frac{c}{\sqrt{a^2+b^2}} \qquad \ldots(2)$$

Case I. If c is positive, then the R.H.S. of (2) is negative. So, we have to make it positive by changing the signs in (2) throughout, for "p" appearing on the R.H.S. of the Normal form is *always* positive.

∴ Changing the signs of (2) throughout, we get

$$-\frac{a}{\sqrt{a^2+b^2}}\,x-\frac{b}{\sqrt{a^2+b^2}}\,y=\frac{c}{\sqrt{a^2+b^2}}$$

which gives the Normal form

$$\left\{\textit{Check}:\ (\text{co-efft. of } x)^2+(\text{co-efft. of } y)^2\right.$$

$$\left.=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1\right\}.$$

Case II. If c is negative, then the R.H.S. of (2) is positive. Therefore, (2) represents in itself the Normal form of the straight line, and we need not change the signs throughout as was done in Case I.

Hence we give the following working rule for reducing the general equation of a straight line to the perpendicular (Normal) form :—

(*i*) *Divide throughout by*

$$\sqrt{(\textit{co-efft. of } x)^2+(\textit{co-efft. of } y)^2}$$

(*ii*) *Transpose the constant term to the R.H.S., and make it positive (by changing the signs throughout) if necessary.*

Examples : Reduce the equations

(1) $3x+4y-15=0$ (2) $\sqrt{3}x-y+2=0$

to the perpendicular form.

Sol. (1) $3x+4y-15=0.$

Dividing throughout by $\sqrt{(3)^2+(4)^2}$, *i.e.*, by 5, we get

$$\tfrac{3}{5}x+\tfrac{4}{5}y-3=0.$$

Transposing the constant term to the R.H.S. we have

$$\tfrac{3}{5}x+\tfrac{4}{5}y=3 \qquad \ldots(i)$$

which gives the required form.

(2) $\sqrt{3}\,x-y+2=0.$

Dividing throughout by $\sqrt{(\sqrt{3})^2+(-1)^2}$ *i.e.*, by 2, we have

$$\frac{\sqrt{3}}{2}x-\frac{1}{2}y+1=0$$

or $$\frac{\sqrt{3}}{2}x-\frac{1}{2}y=-1.$$

Here the constant term on the R.H.S. is −ve, and we have to make it +ve. Therefore, changing the signs throughout, we get

$$-\frac{\sqrt{3}}{2}x+\frac{1}{2}y=1 \qquad \ldots(ii)$$

which gives the required form.

Note : (1) In the transformed equation (i),

$$\cos\alpha=\tfrac{3}{5},\quad \sin\alpha=\tfrac{4}{5},\quad p=3$$

and in (ii) $\cos\alpha=-\frac{\sqrt{3}}{2}$, $\sin\alpha=\frac{1}{2}$

$$p=1.$$

Note : (2) *The student is advised to carefully study Article 3.13 ; it is not as easy as Article 3.9 or Article 3.11.*

Solved Examples

Ex. 1. Find the equation of the straight line which makes an angle of 135° with the axis of x and cuts the axis of y at a distance −3 from the origin.

Sol. Here $m=\tan 135°=-\cot 45°=-1$

and $$c=-3$$

∴ $$y=mx+c \text{ gives}$$
$$y=-x-3$$

or $$x+y+3=0$$

which is the required equation.

Ex. 2. Find the equation of a straight line which passes through the point (2, 3) and makes equal intercepts on the axes.

Sol. Let "a" be the length of each intercept. Therefore, the equation of the straight line (in the intercept form) is

$$\frac{x}{a}+\frac{y}{a}=1$$

or $$x+y=a \qquad \ldots(1)$$

Now (1) passes through (2, 3)

$\therefore \qquad 2+3=a$ which gives $a=5$.

Hence (1) becomes

$$x+y=5$$

which is the required equation.

Ex. 3. Find the equation of the line for which :—

$$p=2, \quad \sin\alpha=\tfrac{4}{5}.$$

Sol. Here $\sin\alpha=\frac{4}{5}$

$\therefore \qquad \cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$.

Hence, $\qquad x\cos\alpha+y\sin\alpha=p$ gives

$$\tfrac{3}{5}x+\tfrac{4}{5}y=2$$

which is the required equation.

Ex. 4. Reduce the equation $3x-4y+12=0$ to (*i*) the slope form (*ii*) the intercept, and (*iii*) the perpendicular form.

Sol. (*i*) $\qquad 3x-4y+12=0$

can be written as $\qquad -4y=-3x-12$

or $\qquad y=\frac{3}{4}x+3$

which gives the slope form with $\frac{3}{4}$ as the slope and 3 as the intercept on the y-axis.

(*ii*) $\qquad 3x-4y+12=0$ can be written as

$$3x-4y=-12$$

or $\qquad \dfrac{3x}{-12}-\dfrac{4y}{-12}=1 \qquad$ (Please note this step)

or $\qquad \dfrac{x}{-4}+\dfrac{y}{3}=1$

which gives the intercept form of the line with -4 and 3 as the intercepts on the axis of x and y respectively.

(*iii*) $$3x-4y+12=0.$$

Dividing throughout by $\sqrt{(\text{co-eff. } x)^2+(\text{co-eff. } y)^2}$ *i.e.*, by $\sqrt{9+16}=5$, we get

$$\tfrac{3}{5}x-\tfrac{4}{5}y+\tfrac{12}{5}=0$$

or $$\tfrac{3}{5}x-\tfrac{4}{5}y=-\tfrac{12}{5}.$$

Now the constant term on the R.H.S. is to be made +ve. Therefore, changing signs throughout, we get

$$-\tfrac{3}{5}x+\tfrac{4}{5}y=\tfrac{12}{5}$$

which gives the required perpendicular form of the straight line with $\sin\alpha=\frac{4}{5}$, $\cos\alpha=-\frac{3}{5}$, and $p=\frac{12}{5}$.

Exercise 5

Find the equation to the straight line :—

1. Cutting off an intercept equal to unity from the positive direction of the y-axis and inclined at an angle of 45° to the x-axis.

2. Cutting off an intercept -5 from the axis of y and being equally inclined to the axes.

Easy

3. Cutting off an intercept 2 from the negative direction of the y-axis and inclined at 30° to the x-axis.

4. Cutting off an intercept -3 from the y-axis and inclined at an angle $\tan^{-1}\frac{3}{5}$ to the axis of x.

Find the equation to the straight line.

5. Cutting off intercepts 3 and 2 from the axes.

6. Cutting off intercepts -5 and 6 from the axes.

7. Find the equation to the st. line which passes through the point (5, 6) and has intercepts on the axes

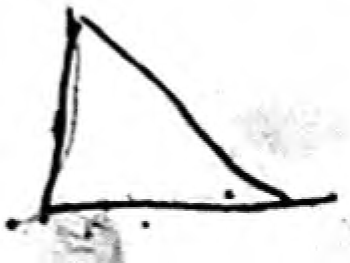
(*i*) equal in magnitude and both positive

(*ii*) equal in magnitude but opposite in sign.

Imp.

8. Find the equation of the straight line which passes through (x', y') and is such that the given point bisects the part intercepted between the axes.

V. V. Imp.



9. Find the equation of the straight line which passes through the point $(-4, 3)$ and is such that the portion of it between the axes is divided by the point in the ratio $5:3$.

10. Find the equation of the straight line which passes through $(1, -1)$ and the sum of whose intercepts on the co-ordinate axes is 8.

11. Find the equation of the straight line which makes intercepts on the axes twice as long as those made by the line

$$7y-6x=42$$

12. What angle do the following lines with the x-axis:—

(*i*) $x+y=1$ (*ii*) $x-y+7=0$

(*iii*) $\sqrt{3}\,x-y+5=0$ (*iv*) $y=\frac{1}{\sqrt{3}}x+8$

(*v*) $y=3$ (*vi*) $3x+5=0$.

Find the equations of the lines for which

13. $p=2, \alpha=60°$ 14. $p=5, \alpha=135°$

15. $p=2, \alpha=150°$ 16. $p=2, \sin\alpha=\frac{4}{5}$.

17. Transform the following to the Normal form :—

(*i*) $\sqrt{3}\,x+y-9=0$ (*ii*) $\sqrt{3}\,x-y+7=0$

(*iii*) $\sqrt{3}\,x+y+5=0$ (*iv*) $\sqrt{3}\,x-y-11=0$

18. Reduce the following equations to the slope form :—

(*i*) $2x-3y+6=0$ (*ii*) $x+2y-3=0$

(*iii*) $x\cos\alpha+y\sin\alpha=p$.

19. Reduce the following equations to the intercept form :—

(*i*) $3x-4y+5=0$ (*ii*) $4x=5y-20$

(*iii*) $x\cos\alpha+y\sin\alpha=\sin 2\alpha$.

(*iv*) $3x+y=9$ (*v*) $2x-8y=16$.

20. Trace the straight lines whose equations are :—

(*i*) $4x=5y-20$ (*ii*) $4x-20=5y$

(*iii*) $4x+5y=20$ (*iv*) $4x+5y+20=0$

Other Forms

3.14. One-Point form of a Straight Line

To find the equation of a straight line passing through a given point (x', y') *and whose slope is* m.

Let the equation of the line be

$$y=mx+c \qquad \ldots(1)$$

If it passes through (x', y'), then

$$y'=mx'+c \qquad \ldots(2)$$

Subtracting (2) from (1), we have

$$\mathbf{y-y'=m(x-x')}$$

which is the required equation.

Note : (1) *The above form is known as* **Point-slope** *form as well.*

Note : (2) *The student is advised to fully understand the significance of the above form. This form of the straight line is of utmost importance so far as the solution of questions on the straight line is concerned.*

3.15. Two-Point form of a Straight Line

To find the equation of a straight line passing through two given paints (x_1, y_1) *and* (x_2, y_2).

First Method : *(To be preferred).*

The equation of any line passing through (x_1, y_1) is

$$y-y_1=m(x-x_1) \qquad \ldots(1) \qquad \text{(Art. 3.15)}$$

If it passes through (x_2, y_2), then

$$y_2-y_1=m(x_2-x_1)$$

$$\therefore \qquad m=\frac{y_2-y_1}{x_2-x_1}$$

Substituting this value of m in (1), we have

$$\mathbf{y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)}$$

which is the required equation.

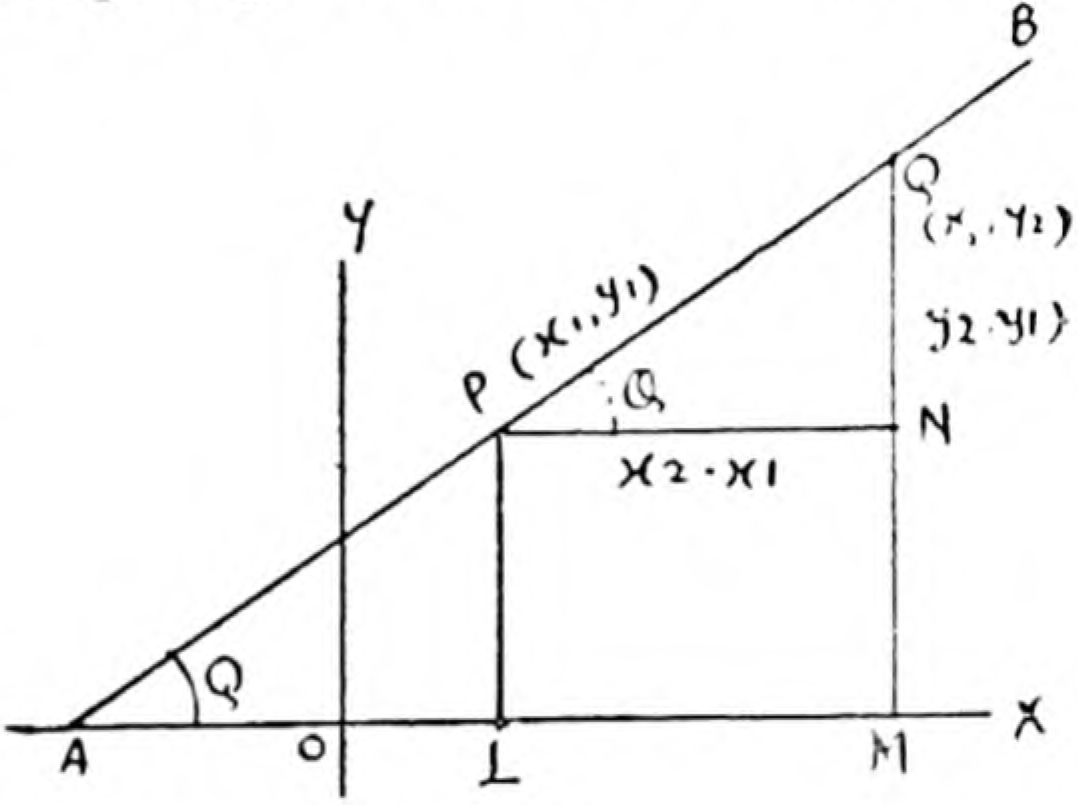
An Important Note

The **slope** of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is

$$\mathbf{m}=\frac{\mathbf{y_2-y_1}}{\mathbf{x_2-x_1}}$$

This can be geometrically proved as under :—

Let AB beast. line making angle θ with the x-axis. Let $P(x_1, y_1)$ and $Q\,(x_2, y_2)$ be the two points through which the line passes. Draw PL, QM perpendiculars on OX and PN perpendicular on QM. Then $\angle QPN=\theta$



Now $PN=LM$

$=OM-OL=x_2-x_1$

and $QN=QM-MN$

$=QM-PL=y_2-y_1$

Hence from the right-angled triangle QPN,

we have $\mathbf{m}=\mathbf{tan}\ \theta=\dfrac{\mathbf{QN}}{\mathbf{PN}}=\dfrac{\mathbf{y_2-y_1}}{\mathbf{x_2-x_1}}$

Second Method :

Let the equation of the line be

$$y=mx+c \qquad \text{...(1)}$$

Since it passes through (x_1, y_1) and (x_2, y_2) we have

$$y_1=mx_1+c \qquad \text{...(2)}$$

and $$y_2=mx_2+c \qquad \text{...(3)}$$

Subtracting (2) from (1), and (2) from (3), we obtain

$$y-y_1=m(x-x_1) \qquad \text{...(4)}$$

and $$y_2-y_1=m(x_2-x_1) \qquad \text{...(5)}$$

Dividing (4) by (5), we get

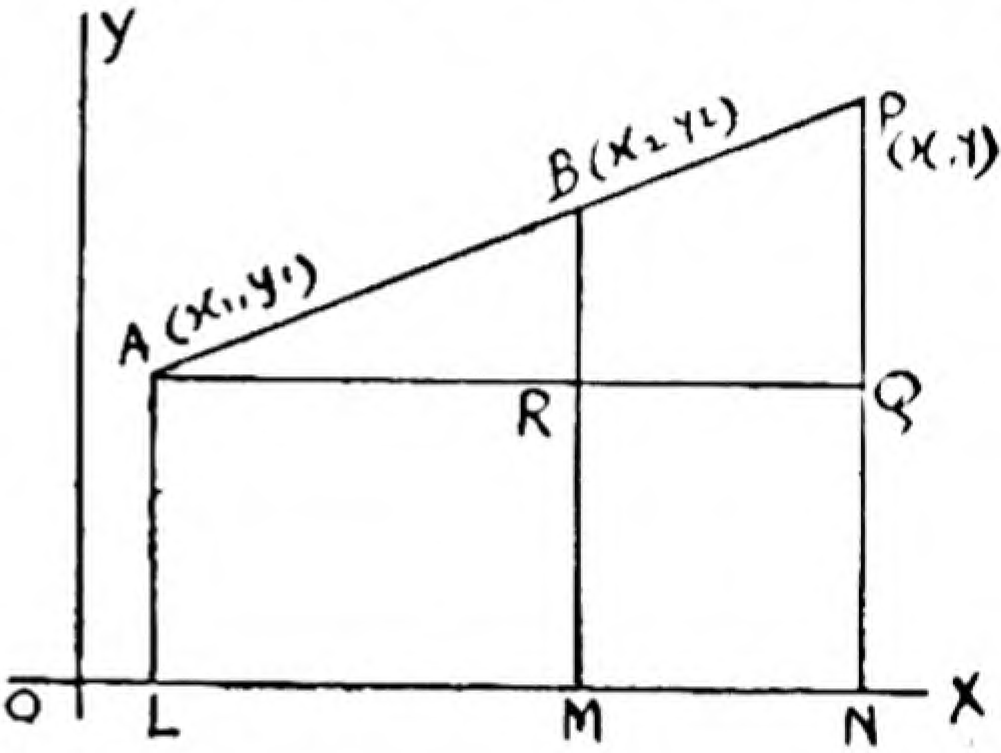
$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

or $$\mathbf{y}-\mathbf{y}_1=\frac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}(\mathbf{x}-\mathbf{x}_1)$$

which is the required form.

Third Method : (*Independent one*)

Let P(x, y) be any any other point on the line AB where the co-ordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively.



Draw AL, BM and PN perpendiculars on OX and AQ perpendicular on PN meeting BM in R. Now by parallels

$$\frac{QP}{RB}=\frac{QA}{AR}$$

or $$\frac{PN-NQ}{MB-MR}=\frac{LN}{LM}$$

or $$\frac{PN-AL}{MB-AL}=\frac{ON-OL}{OM-OL}$$

i.e. $$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

or $$\mathbf{y}-\mathbf{y}_1=\frac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}(\mathbf{x}-\mathbf{x}_1)$$

which is the required form.

3.16. Other Tests for the collinearity of three points.

We have already discussed one test for the collinearity of three points in Article 1.16. That test may be called the *area test*. We will discuss now two more tests which may be called the "**Straight Line Test**" and the "**Slope-Test**".

(*a*) **The Straight Line Test :**

To show that the three points (x_1, y_1) (x_2, y_2) and (x_3, y_3) are collinear, we will first find the equation of the line passing through any two of these three points. After obtaining the equation of the line like this, we will substitute the co-ordinates of the third point in this equation. If this point satisfies the equation, we can say that the three points are collinear (lie in a straight line). *This method is illustrated below :—*

Ex. Show that the three points (2, 3), (−4, 6) and (12, −2) are collinear.

Sol. Equation of the line passing through (2, 3) and (−4, 6) is given by

$$y-3=\frac{6-3}{-4-2}(x-2) \qquad \text{[Article 3.15]}$$

or $$y-3=\frac{3}{-6}(x-2)$$

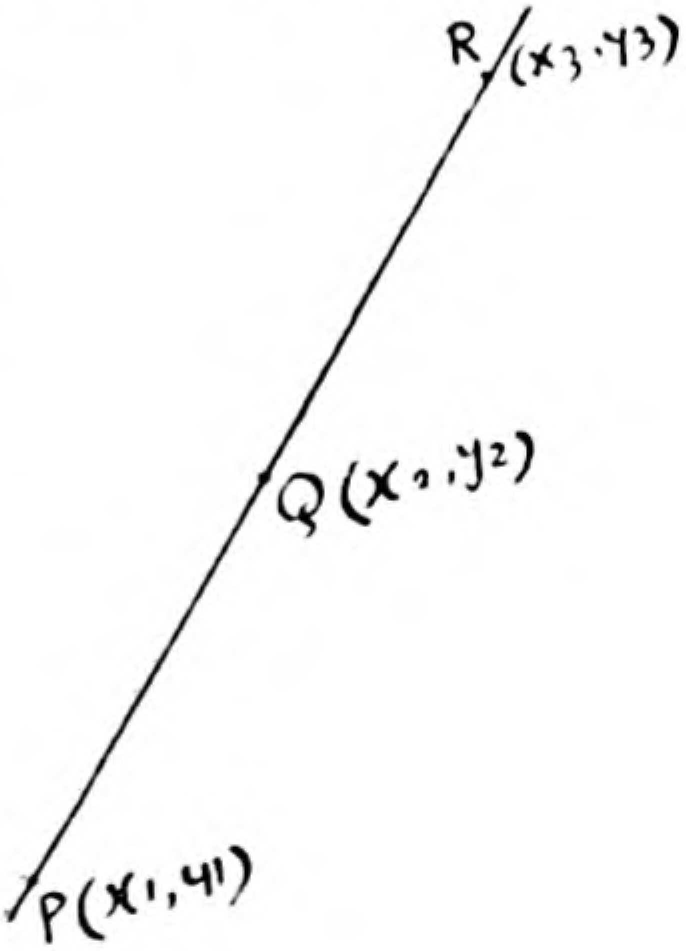
or $$y-3=-\tfrac{1}{2}(x-2)$$

which gives $$x+2y=8. \qquad \ldots(1)$$

Now it is easy for us to check that the third point (12, −2) satisfies (1). Hence the three points are collinear.

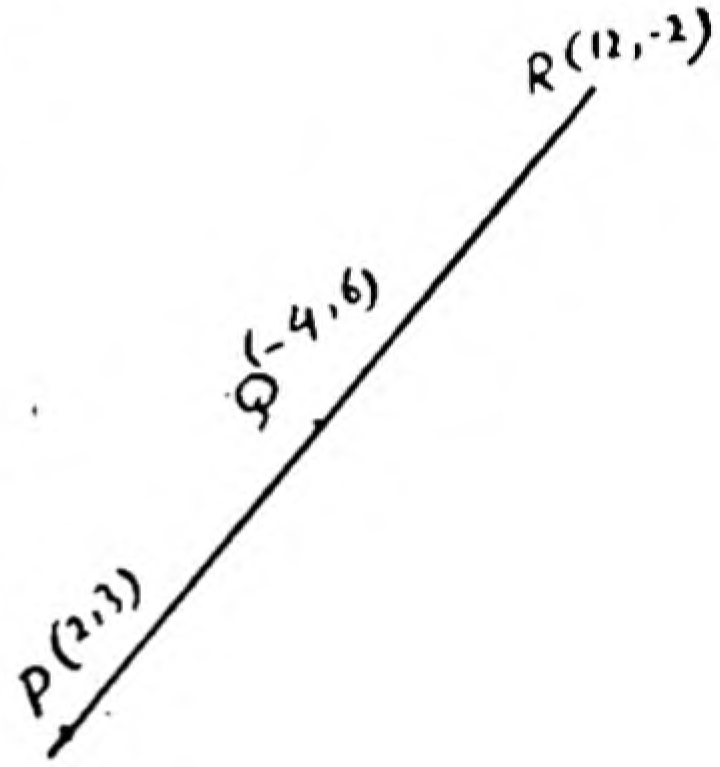
(*b*) **The Slope-Test**

To show that the three points $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_3, y_3)$ are collinear, we will first find the slope of the line passing through any two of these points. Let us do so in the case of PR. Having done this, we will find the slope of the line passing through the 3rd point and anyone of the points already taken with PR. Let us do so for QR. If we find that these slopes are one and the same, we can say that the three points are collinear.



This test is illustrated below :

Ex. Show that the points (2, 3), (−4, 6), and (12, −2) are collinear.



Sol. Slope of the line PR is given by

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-2-3}{12-2}$$

$$=\frac{-5}{10}=-\frac{1}{2} \qquad \ldots(1)$$

Also, slope of the line QR is given by

$$m'=\frac{y_2-y_1}{x_2-x_1}=\frac{6-3}{-4-2}=\frac{3}{-6}=-\frac{1}{2} \qquad \ldots(2)$$

From (1) and (2) we see that

$$m=m'.$$

Hence the three points are collinear.

(∵ *slope cannot be equal unless the three points lie in one and the same line*).

3.17. Distance form of a straight line.

To obtain the equation of a straight line (passing through a given point (x_1, y_1) *and making a given angle* θ *with the x-axis) in the form*

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

when r is the distance of any point (x, y) *from the point* (x_1, y_1).

Let AB be the st. line passing through the given point $Q(x_1, y_1)$ and making an angle $XOB=\theta$. Let $P(x, y)$ be any other point on the line so that $PQ=r$,

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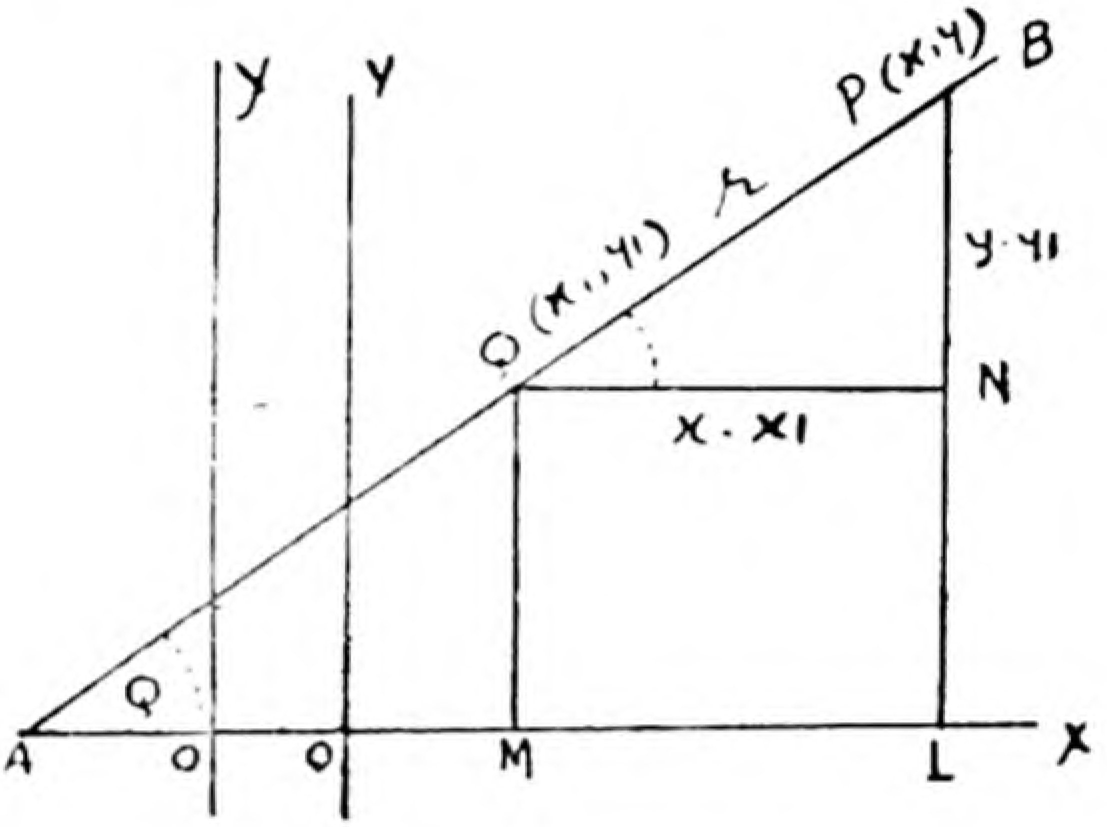
Draw PL, QM perpendiculars on OX and QN perpendicular on PL. Then $\angle PQN = \theta$.



Now from the right-angled $\triangle$ PQN,

$$\text{cor } \theta = \frac{QN}{QP} = \frac{ML}{QP} = \frac{OL-OM}{QP} = \frac{x-x_1}{r}$$

which gives $\dfrac{x-x_1}{\cos\theta} = r$ ...(1)

$$\text{Again, } \sin\theta = \frac{PN}{QP} = \frac{PL-NL}{QP} = \frac{PL-QM}{QP} = \frac{y-y_1}{r}$$

which gives $\dfrac{y-y_1}{\sin\theta} = r$ ...(2)

Hence from (1) and (2), we obtain

$$\mathbf{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r}$$

which is the required equation.

Cor. *The co-ordinates of any point on*

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$$

are given by

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta$$

3.18. Parameter and Parametric Co-ordinates of any point.

Def. *A variable, in terms of which the co-ordinate of any point are expressed, is called a Parameter and the co-ordinates are called the Parametric Co-ordinates.*

Thus $(x_1+r\cos\theta, y_1+r\sin\theta)$ are the parametric co-ordinates of any point on the line

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

and r is the parameter.

Note : (1)

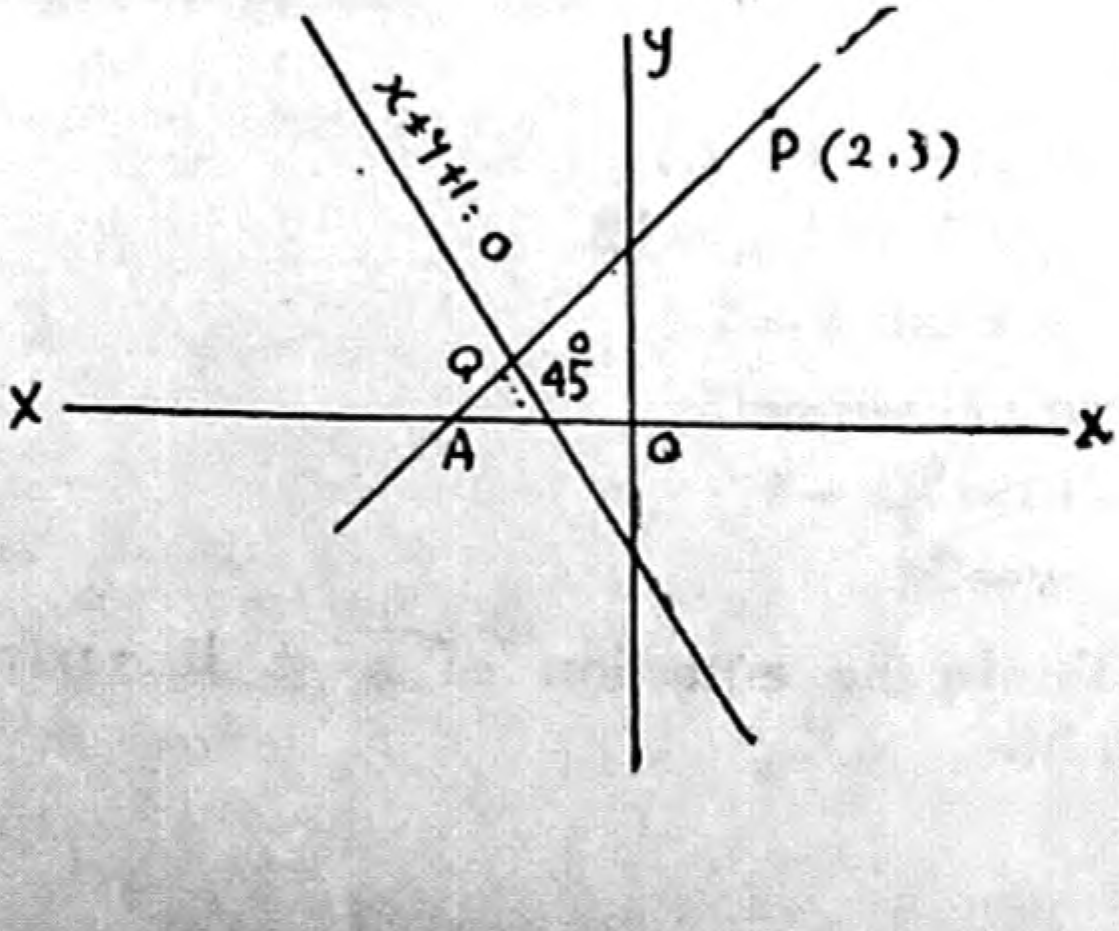
$$x=x_1+r\cos\theta, \qquad y=y_1+r\sin\theta$$

are also called the parametric (or the freedom) equations of the st. line

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

(2) The advantage of the parametric co-ordinates $(x_1+r\cos\theta, y_1+r\sin\theta)$ is that it enables us to find the length "r" of the line from the point (x_1, y_1) to the point where another line meets the given line. *This is illsutrated below* :—

Ex. A st. line making an angle of 45° with the x-axis passes through P(2, 3). Determine the length intercepted on it between the point P and the line $x+y+1=0$.



Let AB be the line passing through P(2, 3) and making 45° with the x-axis. Therefore, its equation is given by

$$\frac{x-2}{\cos 45^\circ}=\frac{y-3}{\sin 45^\circ}=r$$

when r is the distance on the line measured from P to the point Q in which AB is cut by $x+y+1=0$.

Now, co-ordinates of any point on AB are given by

$$(2+r\cos 45^\circ,\ 3+r\sin 45^\circ),\ \textit{i.e.},$$

$$\left(2+\frac{r}{\sqrt{2}},\ 3+\frac{r}{\sqrt{2}}\right).$$

Let these be the co-ordinates of Q

$\therefore$ from $x+y+1=0$, we get

$$\left(2+\frac{r}{\sqrt{2}}\right)+\left(3+\frac{r}{\sqrt{2}}\right)+1=0$$

or $$\frac{2r}{\sqrt{2}}=-6 \text{ or } r=-3\sqrt{2}$$

Changing the —ve sign, we get

$r=3\sqrt{2}$ which is the required length in magnitude.

Solved Examples

Ex. 1. Find the equation of the line which passes through (4, 12) and which makes angle $\tan^{-1}3$ with the x-axis.

Sol. The equation of a line passing through (x_1, y_1) and having slope m is given by

$$y-y_1=m(x-x_1) \qquad \ldots(1)$$

Here $x_1=4$ and $y_1=12$

Also $m=\tan\theta=3$ $\qquad [\because\ \theta=\tan^{-1}3]$

Hence from (1) we get

$$y-12=3(x-4)$$

or $$y=3x.$$

Ex. 2. Obtain the equation of a st. line passing through (2, 3) and (4, 5).

Sol. Here $x_1=2$, $y_1=3$; $x_2=4$, $y_2=5$

$\therefore$ from $y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$, we get

$$y-3=\frac{5-3}{4-2}(x-2)$$

or $$x-y+1=0$$

which is the required equation.

Ex. 3. Show that the points $(a, 0)$, $(0, b)$ and $(3a, -2b)$ lie on a straight line.

Sol. The equation of the line passing through $(a, 0)$ and $(0, b)$ is given by

$$y-0=\frac{b-0}{0-a}(x-a)$$

or $$bx+ay=ab \qquad \ldots(1)$$

Obviously, the third point $(3a, -2b)$ satisfies (1). Hence the three points lie on the straight line whose equation is given by (1).

Ex. 4. Find the equations of the sides of the triangle the co-ordinates of whose angular points are $(0, 1)$, $(2, 0)$ and $(-1, -2)$. Find also the equation of the medians of this triangle.

Sol. Let the angular points be A$(0, 1)$, B$(2, 0)$ and C$(-1, -2)$

$\therefore$ the equation of AB is

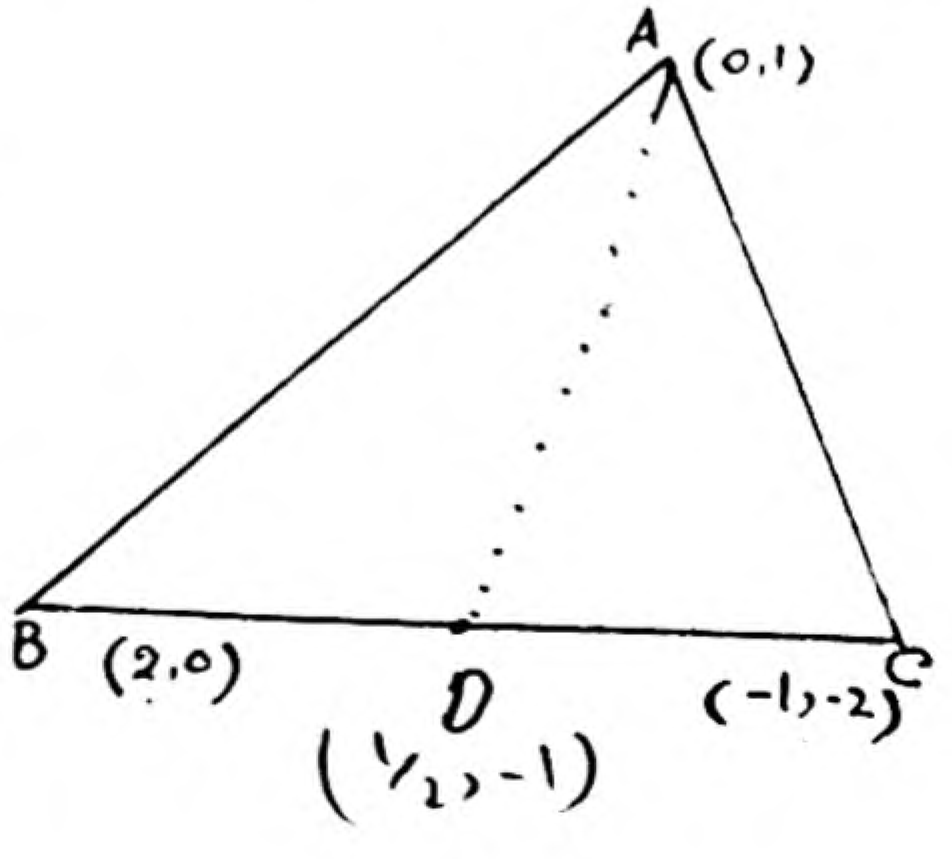
$$y-1=\frac{0-1}{2-0}(x-0)$$

or $$x+2y=2$$

Similarly, the equations of the other two sides are

$2x-3y=4$ and $y-3x=1$

(*ii*) Now the co-ordinates of the middle point of BC $(\frac{1}{2}, -1)$. Hence the equation



of the median AD when (the co-ordinates of A are (0, 1) and that of D are $(\frac{1}{2}, -1)$ is given by

$$y-=\frac{-1-1}{\frac{1}{2}-0}(x-0)$$

or $$y-1=-4(x-0)$$

or $$4x+y=1.$$

The equations of the other two medians can be found in a similar manner, which is left as an exercise for the student.

Exercise 6

1. Find the equation of the line through the point (4, 12) which makes an angle $\tan^{-1} 3$ with the x-axis.

2. Find the equation of a st. line passing through (2, -1) and has a slope -2.

3. Find the equation of a line through (-1, 1) which is equally inclined to the axes.

4. Obtain the equation of a st. line which passes through (0, 0) and is inclined at an angle of (*i*) 45° (*ii*) 35° with the x-axis.

5. Find the equation of a st. line which makes an angle of $90°+\alpha$ with the x-axis and which passes through the point $(p \cos \alpha, p \sin \alpha)$.

6. Prove that the straight line which makes an angle $\tan^{-1} 2$ with the x-axis and which cuts the y-axis in the point (0, -2) passes through the point (1, 0).

7. Find the equation of the line passing through :—

(*i*) (-1, 2) and (2, -1) (*ii*) (0, 0) and (2, -2)

(*iii*) (3, 4) and (5, 6) (*iv*) (-1, 3) and (6, -7)

(*v*) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

(*vi*) $\left(at_1, \frac{a}{t_1}\right)$ and $\left(at_2\ \frac{a}{t_2}\right)$

(*vii*) $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$

(*viii*) $(a \cos \alpha, b \sin \alpha)$ and $(a \cos \beta, b \sin \beta)$

(*ix*) $(a \sec \alpha, b \tan \alpha)$ and $(a \sec \beta, b \tan \beta)$

(*x*) (a, b) and $(a+b, a-b)$

8. Show that the following sets of points are collinear :—

(*i*) $(a, 0)$, $(0, b)$, $(3a, -2b)$

(*ii*) $(2, 3)$, $(-4, 6)$, $(12, -2)$

(*iii*) $(3a, 0)$, $(0, 3b)$, $(a, 2b)$.

9. Find the equations of (*i*) the sides (*ii*) the medians of a triangle whose angular points are $(2, 4)$, $(-4, 1)$ and $(2, -3)$.

10. In what ratio is the line joining the points $(1, 2)$ divided by the line joining the points $(2, 3)$ and $(4, 1)$?

11. Find the equations of the diagonals of the parallelogram the equations of whose sides are :

$$x=a, \ x=b, \ y=c \text{ and } y=d$$

12. Prove that the line $y-x+2=0$ cuts the line joining $(0, -1)$ and $(8, 9)$ in the ratio 2 : 3.

13. Find the co-ordinates of a point which is $2\sqrt{2}$ units from the point $(3, 4)$ and lies on a straight line passing through this point and is inclined at an angle of $\frac{3\pi}{4}$ to the x-axis. (*P.U.*)

14. Find the distance of the point $(2, 3)$ from the line $2x-y+9=0$ measured along a line which makes an angle of 45° with the x-axis. Answer :- $-10\sqrt{2}$

15. Find the equation of the line through the point A$(2, 3)$ inclined at an angle of 30° to the x-axis. Determine the length intercepted on it between the point A and *the line* $2x+3y+4=0$.

16. Find the direction in which a st. line must be drawn through the point $(1, 2)$ so that its point of intersection with $x+y=4$ may be at a distance $\frac{1}{3}\sqrt{6}$ from this point.

CHAPTER IV

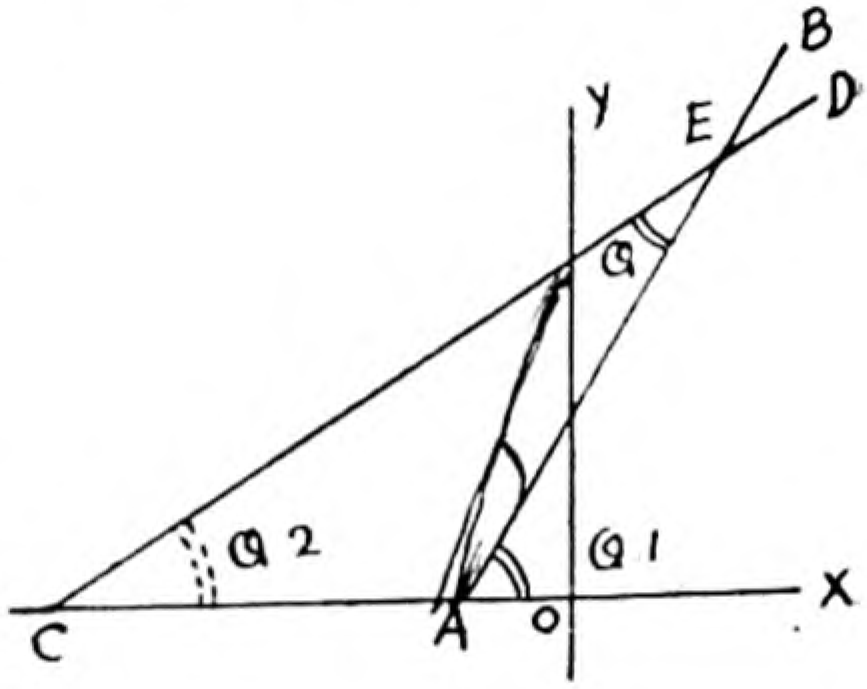
Q

ANGLE BETWEEN TWO STRAIGHT LINES

4.1. (a) *To find the angle between the two straight lines*

$$y=m_1x+c_1 \text{ and } y=m_2x+c_2,$$

Let AB, CD be the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ making angles θ_1 and θ_2 with the x-axis respectively, so that $\tan \theta_1=m_1$ and $\tan \theta_2=m_2$.



Let θ be the angle between these lines so that

$$\theta=\theta_1-\theta_2$$

$$(\because \quad \theta_1=\theta_2+\theta)$$

$$\therefore \quad \tan \theta=\tan (\theta_1-\theta_2)$$

$$=\frac{\tan \theta_1-\tan \theta_2}{1+\tan \theta_1 \tan \theta_2}$$

$$\text{or} \quad \tan \theta=\frac{m_1-m_2}{1+m_1m_2}$$

$$\therefore \quad \boldsymbol{\theta=\tan^{-1} \frac{m_1-m_2}{1+m_1m_2}}$$

which gives the required angle.

Note : *If tan* $\theta=\frac{m_1-m_2}{1+m_1m_2}$ *gives a positive result, then tan* θ *is the tangent of the* **acute** *angle between the lines* $y=m_1x+c_1$ *and* $y=m_2x+c_2$; *and if tan* $\theta=\frac{m_1-m_2}{1+m_1m_2}$ *gives a* **negative** *result, then tan* θ *is the tangent of the* **obtuse** *angle between these lines.*

(*b*) *To find the angle between the two straight lines*

$$a_1x+b_1y+c_1=0 \quad and \quad a_2x+b_2y+c_2=0.$$

Let m_1 and m_2 be the slopes of these two lines.

$$\therefore \qquad m_1=-\frac{a_1}{b_1} \text{ and } m_2=-\frac{a_2}{b_2}.$$

Hence from the formula

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}, \text{ we get}$$

$$\tan\theta=\frac{-\frac{a_1}{b_1}+\frac{a_2}{b_2}}{1+\frac{a_1}{b_1}\times\frac{a_2}{b_2}}=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}$$

which gives

$$\theta=\tan^{-1}\frac{\mathbf{a_2b_1-a_1b_2}}{\mathbf{a_1a_2+b_1b_2}}.$$

Note : *Failure of the angle-formula*

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}.$$

The above formula fails if one of the lines is of the form $x=a$ because the slope of such a line is *infinite.* In this case, we make use of the second form of the angle-formula *i.e.*,

$$tan\ \theta=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

4.2. Condition of parallelism

(*a*) *Condition under which the lines* $y=m_1x+c_1$ *and* $y=m_2x+c_2$ *can be parallel.*

The two lines will be parallel if the angles at which these are inclined to the x-axis are equal.

Let these angles be θ_1 and θ_2.

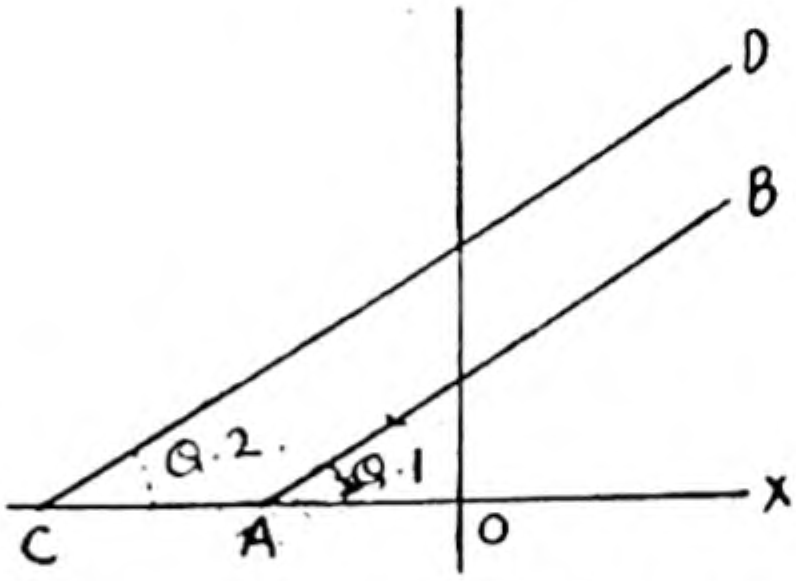
Then

$$\theta_1 = \theta_2$$

or $$\tan\theta_1 = \tan\theta_2$$

$\therefore$ $$\mathbf{m_1 = m_2}$$

$$[\because \quad \tan\theta_1 = m_1 \text{ etc.}]$$



Hence the two st. lines $y = m_1x + c_1$ *and* $y = m_2x + c_2$ *will be parallel if their slopes are equal.*

This can be proved even as under :—

The two straight lines will be parallel if they do not meet, i.e., $\theta = 0$. *Thus in such a case tan* $\theta = 0$ which gives

$$\frac{m_1 - m_2}{1 + m_1m_2} = 0$$

or $$\mathbf{m_1 = m_2}.$$

(*b*) *Condition under which the lines*

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0$$

can be parallel.

Here $m_1 = -\dfrac{a_1}{b_1}$ and $m_2 = -\dfrac{a_2}{b_2}$

Now the lines will be parallel if (already proved)

$$m_1 = m_2$$

i.e., $$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

which gives

$$\frac{\mathbf{a_1}}{\mathbf{a_2}} = \frac{\mathbf{b_1}}{\mathbf{b_2}}$$

as the required condition.

Cor. *Equation of a straight line parallel to the given line*

$$ax + by + c = 0.$$

Slope of the given line$=-\frac{a}{b}$

$\therefore$ Equation of the required line is

$$y=-\frac{a}{b}x+k$$

where k is a constant. This when simplified gives

$$ax+by+d=0 \qquad \text{where } d=-bk$$

Hence we give the following rule to write down the equation of any line parallel to a given line (whose equation is given in the general form).

In the equation of the given line, change only the constant to some other constant which can be found from the second condition.

Illustration : Find the equation of the straight line parallel to $2x+5y+2=0$ and passing through the point $(1, 1)$.

Sol. $2x+5y+k=0$ will be parallel to the line

$$2x+5y+2=0$$

Now $\quad 2x+5y+k=0$ passes through $(1, 1)$.

$\therefore \quad 2\times 1+5\times 1+k=0$

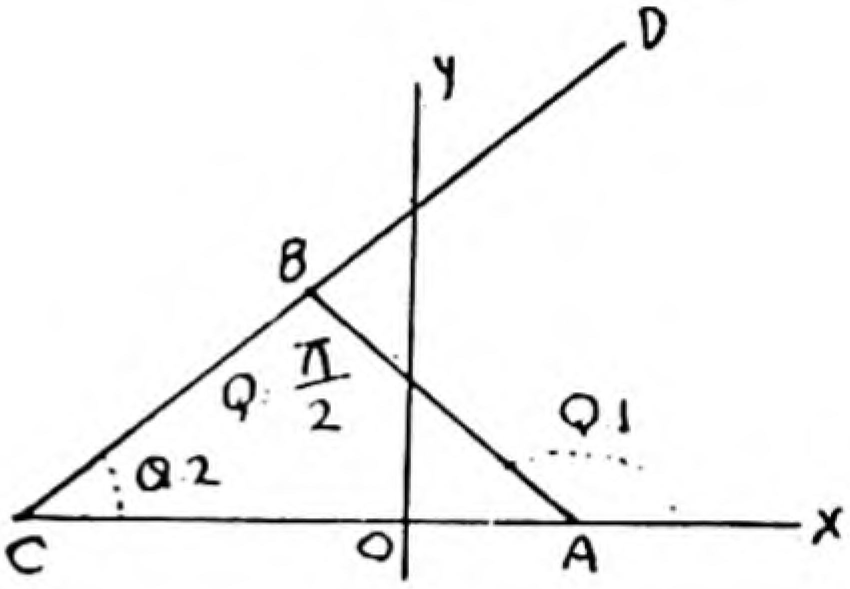
or $\quad k=-7.$

Hence the equation of the required line is

$$2x+5y-7=0$$

4.3. Condition of Perpendicularity

(a) *To find the condition under which the two lines* $y=m_1x+c_1$ *and* $y=m_2x+c_2$ *can be perpendicular to each other.*



Let AB, CD be the lines

$$y=m_1x+c_1 \text{ and}$$
$$y=m_2x+c_2 \text{ and}$$

θ_1 and θ_2 be the angles at which these are inclined to the x-axis, so that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.

Now the two lines will be perpendicular

$$\text{if } \theta_1=\theta_2+\frac{\pi}{2}.$$

$$\text{or} \quad \tan\theta_1=\tan\left(\theta_2+\frac{\pi}{2}\right)=\cot\theta_2$$

$$\text{or} \quad \tan\theta_1=-\frac{1}{\tan\theta_2}$$

$$\text{or} \quad m_1=-\frac{1}{m_2}$$

which gives $\mathbf{m_1\, m_2=-1}$

Hence the two st. lines will be perpendicular if the product of their slopes is equal to -1.

This can be proved even as under :—

The two lines will be perpendicular if $\theta=90$, *i.e., if cot* $\theta=0$

Thus from the formula

$$\cot\theta=\frac{1+m_1m_2}{m_1-m_2}$$

we get

$$0=\frac{1+m_1m_2}{m_1-m_2}$$

This gives

$$m_1m_2+1=0$$

$$\text{or} \quad \mathbf{m_1m_2=-1}$$

(*b*) *Condition under which the two lines* $a_1x+b_1y+c_1=0$ *and* $a_2x+b_2y+c_2=0$ *can be perpendicular.*

Let m_1 and m_2 be the slopes of these lines. Then

$$m_1=-\frac{a_1}{b_1} \text{ and } m_2=-\frac{a_2}{b_2}$$

Now in order that the lines may be perpendicular

$$m_1\,m_2=-1$$

$$\therefore \qquad \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)=-1$$

or

$$\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}=-1$$

or

$$\mathbf{a_1a_2+b_1b_2=0}$$

which gives the required condition.

Cor. *Equation of a straight line perpendicular to the given line* $ax+by+c=0$

Slope of the given line $=-\dfrac{a}{b}$

Let m be the slope of the required line.

$$\therefore \qquad m\left(-\frac{a}{b}\right)=-1 \qquad (\because \; m_1\,m_2=-1)$$

This gives $m=\dfrac{b}{a}$

$\therefore$ the equation of the required line is

$$y=\frac{b}{a}x+c \qquad \ldots(1)$$

on simplification (1) can be written as

$$bx-ay+k=0 \qquad \text{where } k=ac$$

Hence we give the following rule to write down the equation of any line perpendicular to a given line (*whose equation is given in the general form*).

In the equation of the given line (*i*) interchange the co-efficients of x and y (*ii*) change the sign of one of them and (*iii*) change the constant term to a new constant k, which can be found from the second condition.

Illustration : Find the equation of a straight line through the given point (4, 5) and perpendicular to the line

$$3x-2y+5=0$$

The equation of any line perpendicular to the given line is

$$2x+3y+k=0 \qquad \text{...(1)}$$

Now (1) passes through (4, 5)

$\therefore \qquad 2\times4+3\times5+k=0$

which gives $\qquad k=-23$

Hence (1) becomes

$$2x+3y-23=0.$$

which is the required line.

Imp. Art.

4.4. To find the equation of a straight line passing through the point ($\mathbf{x_1}$, $\mathbf{y_1}$) and (*i*) parallel (*ii*) perpendicular to the given line $\mathbf{ax+by+c=0}$.

The equation of any st. line passing through the point (x_1, y_1) and having slope m is

$$y-y_1=m(x-x_1) \qquad \text{...(1)} \qquad \text{(Art. 3.14)}$$

(*a*) Now if (1) is parallel to the given line,

$$m=-\frac{a}{b} \qquad (m_1=m_2)$$

$\therefore$ in such a case (1) becomes

$$\mathbf{y-y_1=-\frac{a}{b}(x-x_1)} \qquad \text{...(A)}$$

(*b*) Again, if (1) is perpendicular to the given line, then

$$m=\frac{b}{a} \qquad \left(m_1=-\frac{1}{m_2}\right)$$

∴ in such a case it becomes

$$\mathbf{y}-\mathbf{y}_1=\frac{\mathbf{b}}{\mathbf{a}}(\mathbf{x}-\mathbf{x}_1) \qquad \ldots(B)$$

Note : *The student is advised to commit the results (A) and (B) to memory. These are of great importance and are preferable to the rules given in the corollaries of Articles* 4.2 *and* 4.3.

4.5. Determination of a straight line

Let $ax+by+c=0$ be the equation of a straight line. This can be put as

$$y=-\frac{a}{b}x-\frac{c}{b} \qquad \ldots(1)$$

Equation (1) shows that only two constants appear in the equation of a straight line. These are $-\frac{a}{b}$ and $-\frac{c}{b}$ and can be determined from any two given conditions. These conditions can be different for different questions. *Hence, any two given oonditions are sufficient for us to find the equation of a straight line.* This has been illustrated below by means of some solved examples.

Solved Examples

Ex. (1) Find the equation of a straight line through the point (3, 4) and parallel to the line $2x+5y+7=0$

Sol.

First Method : (*K-method*) (Best method ·

The given line and the required line being parallel must have their slopes equal. Therefore, the equation of the required line can be put as

$$2x+5y+k=0 \qquad \ldots(1) \qquad \text{(Cor. of Art. 4.2)}$$

Now (1) passes through (3, 4)

∴ $2\times3+5\times4+k=0$

or $k=-26$

Hence (1) gives

$$2x+5y=26$$

which gives the required equation.

Second Method : (*Slope-method*)

Let $y=mx+c$...(1) be the equation of the required line.

This is parallel to the line

$$2x+5y+7=0$$

$$\therefore \qquad m=-\frac{2}{5}$$

Thus (1) becomes

$$y=-\frac{2}{5}x+c \qquad \text{...(2)}$$

Also, this passes through (3, 4)

$$\therefore \qquad 4=-\frac{2}{5}\times 3+c \text{ which gives}$$

$$c=\frac{26}{5}$$

Hence (2) becomes

$$y=-\frac{2}{5}x+\frac{26}{5}$$

or $$2x+5y=26$$

Third Method : (*One-point form method to be preferred*)

The equation of a straight line passing through the point (3, 4) is given by

$$y-4=m(x-3) \qquad \text{...(1)}$$

where m is its slope. Now that (1) is parallel to the given line, we have

$$m=-\frac{2}{5}$$

Hence it becomes

$$y-4=-\frac{2}{5}(x-3)$$

or $$2x+5y=26$$

Ex. 2. Find the equation of a straight line passing through (2, −3) and perpendicular to the given line

$$2x+3y+1=0$$

Sol. 1st Method : (*k-method*)

The equation of the required line can be put as

$$3x-2y+k=0 \qquad \ldots(1)$$

(Cor. of Art. 4.3)

Now (1) passes through (2, −3)

$$\therefore \quad 3\times 2-2\times -3+k=0$$

or $$k=-12$$

Hence (1) gives

$$3x-2y-12=0$$

which is the equation of the required line.

Second Method : (*Slope method*)

Let the equation of the required line be

$$y=mx+c \qquad \ldots(1)$$

Now the slope of the given line $=-\frac{2}{3}$

$\therefore$ the slope of the required line, *i.e.*, $m=\frac{3}{2}$

$$\left(\because \ m_1=-\frac{1}{m_2}\right)$$

Thus (1) becomes

$$y=\tfrac{3}{2}x+c \qquad \ldots(2)$$

Again (2) passes through (2, −3)

$$\therefore \quad -3=\tfrac{3}{2}\times 2+c \quad \text{or} \quad c=-6$$

Hence (2) becomes

$$y=\tfrac{3}{2}x-6$$

or $$3x-2y-12=0$$

Third Method : (*One-point form method to be preferred*)

The equation of a straight line passing through the point (2, -3) is of the form

$$y+3=m(x-2) \qquad \ldots(1)$$

when m is its slope. Now that the required line is perpendicular to the given line, we have

$$m=\tfrac{3}{2} \qquad \left(\because \ m_1=-\frac{1}{m_1}\right)$$

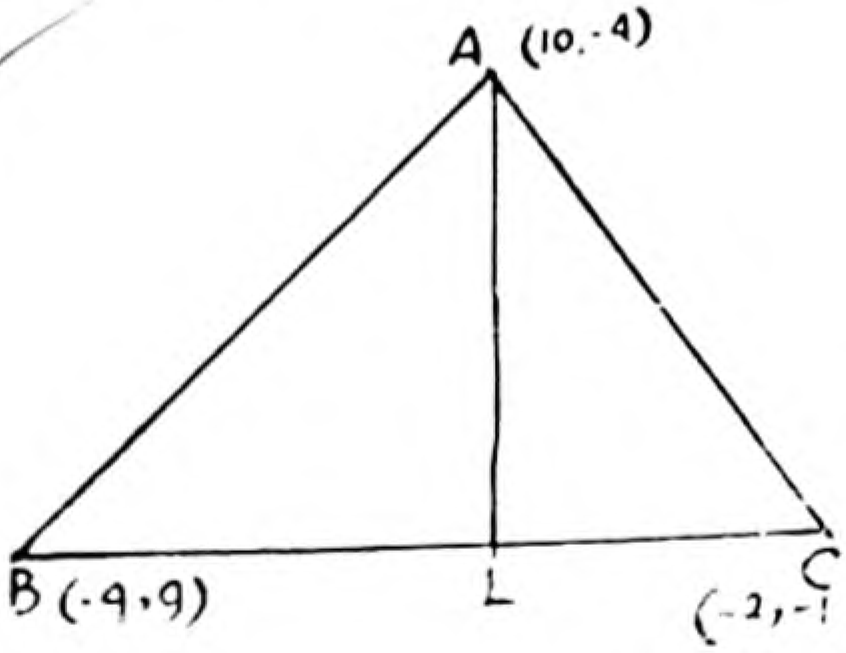
Hence (1) becomes

$$y+3=\tfrac{3}{2}(x-2)$$

or $$3x-2y-12=0$$

Ex. 3. Given the triangle A(10, 4), B(-4, 9), C(-2, -1), find the equation of the altitude through A.

Sol. Let AL be the required altitude. This is a straight line through (10, 4) and perpendicular to the side BC. Now the slope of the line BC is given by



$$\frac{y_2-y_1}{x_2-x_1}=\frac{-1-9}{-2+4}=-5=m'$$

$\therefore$ the slope of AL $=\tfrac{1}{5}=m$

$$\left(\because \ m=-\frac{1}{m'}\right)$$

Hence the equation of the altitude is

$$y-4=\tfrac{1}{5}(x-10) \qquad \ldots[y-y_1=m(x-x_1)$$

or $$x-5y+10=0$$

Imp.

Ex. 4. *Find the equations of two straight lines passing through the point (1, −1) and inclined at an angle of 45° to the line 2x − 5y + 7 = 0* **(Very Important)**

Likely to be asked in univ.

Sol. The equation of a st. line passing through the point $(1, -1)$ is of the form

$$y+1=m(x-1) \qquad \ldots(1) \qquad [y-y_1=m(x-x_1)]$$

where m is its slope.

Now the slope of the given line $=\frac{2}{5}$

$$\therefore \quad \tan 45^\circ = \pm \frac{m-\frac{2}{5}}{1+\frac{2m}{5}} \qquad \left(\because \quad \tan\theta = \pm\frac{m_1-m_2}{1+m_1m_2}\right)$$

or $$1=\pm\frac{5m-2}{5+2m}$$

This gives $m=\frac{7}{3}, -\frac{3}{7}$

Substituting $m=\frac{7}{8}$ in (1), we get

$$y+1=\frac{7}{3}(x-1)$$

or $7x-3y=10$

Also, substituting $m=-\frac{3}{7}$ in (1) we get

$$y+1=-\frac{3}{7}(x-1)$$

or $3x+7y+4=0$

Hence $7x-3y=10$

and $3x+7y+4=0$

are the equations of the required line.

Ex. 5. Show that the straight lines $3x+2y+8=0$ and $2x-3y+1=0$ are perpendicular to each other.

Sol. **First Method** :

Let m_1 and m_2 be the slopes of these two lines.

$\therefore \quad m_1=-\frac{2}{2} \quad$ and $\quad m_2=\frac{2}{3}$

Now $m_1m_2=-\frac{3}{2}\times\frac{2}{3}=-1$

Hence the two lines are perpendicular.

Second method :

From the equation of the first line

$a_1=3, \quad b_1=2, \quad$ and

from the equation of the second line

$$a_2=2, \qquad b_2=-3$$

$\therefore \quad a_1a_2+b_1b_2=3\times 2+2\times -3=0$ [Art. 4.3 (*b*)]

Hence the two lines are at right angles to each other.

Ex. 6. Show that the st. lines

$$3x+5y+1=0 \qquad \text{and}$$
$$6x+10y+8=0 \text{ are parallel.}$$

Sol. First Method :

Let m_1 and m_2 be the slopes of these two lines

$\therefore \quad m_1=-\frac{3}{5}$ and $m_2=-\frac{6}{10}=-\frac{3}{5}$

This gives $\qquad m_1=m_2$

Hence the two lines are parallel

Second Method :

From the equation of the first line

$$a_1=3, \qquad b_1=5, \qquad \text{and}$$

from the equation of the second line

$$a_2=6, \qquad b_2=10$$

$$\therefore \qquad \frac{a_1}{a_2}=\frac{3}{6}=\frac{1}{2}, \text{ and } \frac{b_1}{b_2}=\frac{5}{10}=\frac{1}{2}.$$

Hence the two lines are parallel

$$\left(\because \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{2}\right)$$

Exercise 7

Find the angle between the following pairs of lines :—

1. $x-y\sqrt{3}=5$ and $\sqrt{3}x+y=7$.
2. $x-4y=3$ and $6x-y=11$.

3. $y=3x+7$ and $3y-x=8$.

4. Find the tangent of the angle between the lines whose intercepts on the axes are a, $-b$ and b, $-a$ respectively.

5. Prove that the line joining (1, 2) and (2, −2) is perpendicular to the line joining (8, 2) and (4, 1).

6. Find the equation of the perpendicular bisector of the line joining (4, 1) and (2, 3).

Find the equation of the straight line

7. passing through (2, 3) parallel to $3x-4y+5=0$.

8. passing through (4, 5) parallel to $2x-3y-5=0$.

9. passing through (x', y') parallel to $Ax+By+C=0$.

Find the equation of the straight line

10. Passing through the point (2, 3) perpendicular to the line $4x-3y=10$.

11. Passing through the point (−6, 10) and perpendicular to the line $7x+8y=5$.

12. Passing through the point (2, −3) and perpendicular to the line joining the points (5, 7) and (−6, 3).

13. Passing through the point (−4, −3) and perpendicular to the line joining (1, 3) and (2, 7).

N 14. Find the equation of the straight line which bisects the line joining the points (a, b) and (a', b') perpendicularly.

N 15. Find the equation of the right bisector of the line joining (−3, 5) and (2, 1).

V 16. Find the equations of the three altitudes of a triangle whose angular points are (1, 2), (−1, −3), and (5, −1).

17. Find the equations of two lines through the point (4, 5) which make an angle 45° with the line $2x-y+7=0$.

18. Find the equations of the two lines which pass through (3, −2) and are inclined at 60° to the line

$$x\sqrt{3}+y=1.$$

19. Show that the equation of the line passing through the origin and making an angle θ with the line

$$y=mx+c \text{ is } \frac{y}{x}=\frac{m+\tan\theta}{1-m\tan\theta} \qquad (P.U.\ 1951)$$

20. Show that the equation of the line through (h, k) which makes angle α with $y=mx+c$ is

$$y-k=\frac{m+\tan\alpha}{1-m\tan\alpha}(x-h).$$

CHAPTER V

INTERSECTION OF STRAIGHT LINES

5.1. *The co-ordinates of the points of intersection of two straight lines (or any two curves) satisfy the equations of both the straight lines (or the curves).*

This is so because such a point lies on both the lines (or curves).

Hence we give the following general rule for finding the points of intersection of two curves (or straight lines) whose equations are given :—

Solve the two equations simultaneously the same way as we do in Algebra. The corresponding values of x and y will give the co-ordinates of the required point (or points) of intersection.

5.2. *To find the co-ordinates of the point of intersection of two given straight lines whose equations are*

$$a_1x+b_1y+c_1=0 \quad \textit{and}$$

$$a_2x+b_2y+c_2=0.$$

Solving these two equations simultaneously by cross-multiplication, we get

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

These give $x=\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$

and $y=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

Hence $$\left(\frac{\mathbf{b_1c_2-b_2c_1}}{\mathbf{a_1b_2-a_2b_1}},\ \frac{\mathbf{c\,a_2-c_2a_1}}{\mathbf{a_1b_2-a_2b_1}}\right)$$

is the required point of intersection.

Cor. If the two lines are parallel, then

$$\frac{a_1}{a_2}=\frac{b_1}{b_2} \qquad \text{[Article 4.2 (}b\text{)]}$$

or $$a_1b_2-a_2b_1=0.$$

In such a case the co-ordinates of the point of intersection are (∞, ∞).

Hence we can say that two parallel lines meet at infinity.

5 3. Concurrence of three lines

The lines will be concurrent if the point of intersection of any two of them lies on the third, i.e., if the co-ordinates of the point of intersection of two of them satisfy the equation of the third line as well.

Hence we give the following rule to prove that three given lines are concurrent, (i.e., meet in a point).

(*i*) **Solve the equations of two of the lines simultaneously for x and y to get the co-ordinates of their point of intersection.**

(*ii*) **Substitute these co-ordinates in the equation of the third line.**

(*iii*) **If this equation is satisfied by these co-ordinates, the three lines are concurrent.**

5.4. *To find the condition that the three straight lines*

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0,\ a_3x+b_3y+c_3=0$$

may meet in a point.

Solving the first two equations simultaneously as in Article 5.2, we get

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\ y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

Now if the three lines are concurrent then

$$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\quad \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$$

must satisfy the third equation.

$$i.e.,\ a_3\,\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}+b_3\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}+c_3=0$$

which gives

$$\mathbf{a}_3(\mathbf{b}_1\mathbf{c}_2-\mathbf{b}_2\mathbf{c}_1)+\mathbf{b}_3(\mathbf{c}_1\mathbf{a}_2-\mathbf{c}_2\mathbf{a}_1)+\mathbf{c}_3(\mathbf{a}_1\mathbf{b}_2-\mathbf{a}_2\mathbf{b}_1)=0$$

as the required condition

Alternative Method :—

Let (x_1, y_1) be the point through which all the three lines pass.

Then $$a_1x_1+b_1y_1+c_1=0 \qquad ...(1)$$
$$a_2x_1+b_2y_1+c_2=0 \qquad ...(2)$$
$$a_3x_1+b_3y_1+c_3=0 \qquad ...(3)$$

Eliminating x_1, y_1 from (1) (2) and (3), we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which gives the required condition.

5.5. Special Test for the concurrence of three lines

If three non-zero constants l, m, n can be found such that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0$$

identically, *then the three lines*

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0,\ a_3x+b_3y+c_3=0$$

are concurrent.

Let us suppose that some point (x_1, y_1) lies on the first two lines, then

$$a_1x_1+b_1y_1+c_1=0$$

and $$a_2x_1+b_2y_1+c_2=0$$

$\therefore$ from

$$l(a_1x_1+b_1y_1+c_1)+m(a_2x_1+b_2y_1+c_2)+n(a_3x_1+b_3y_1+c_3)=0$$

we get

$$l(0)+m(0)+n(a_3x_1+b_3y_1+c_3)=0$$

or $$n(a_3x_1+b_3y_1+c_3)=0$$

Now $$n\neq 0$$

$\therefore$ $$a_3x+b_3y_1+c_3=0$$

which shows that the third line

$$a_3x+b_3y+c_3=0$$

also passes through (x_1, y_1).

Hence the three straight lines are concurrent.

Note : The above test is called the l, m, n test and should not be applied in the case of the equations of straight lines which have numercial co-efficients. In all such cases, the method explained in Article 5.4 should be applied. That will enable the student to find the co-ordinates of the point of concurrence also.

Cor. An important particular case

The three straight lines whose equations are

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0,\ a_3x+b_3y+c=0$$

are concurrent, if

$$(a_1x+b_1y+c_1)+(a_2x+b_2y+c_2)+(a_3x+b_3y+c_3)=0 \textit{ identically.}$$

Obviously, $l=m=n=1$ in this case.

Example. *Prove that the altitudes of a triangle are concurrent.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle. Now the slope

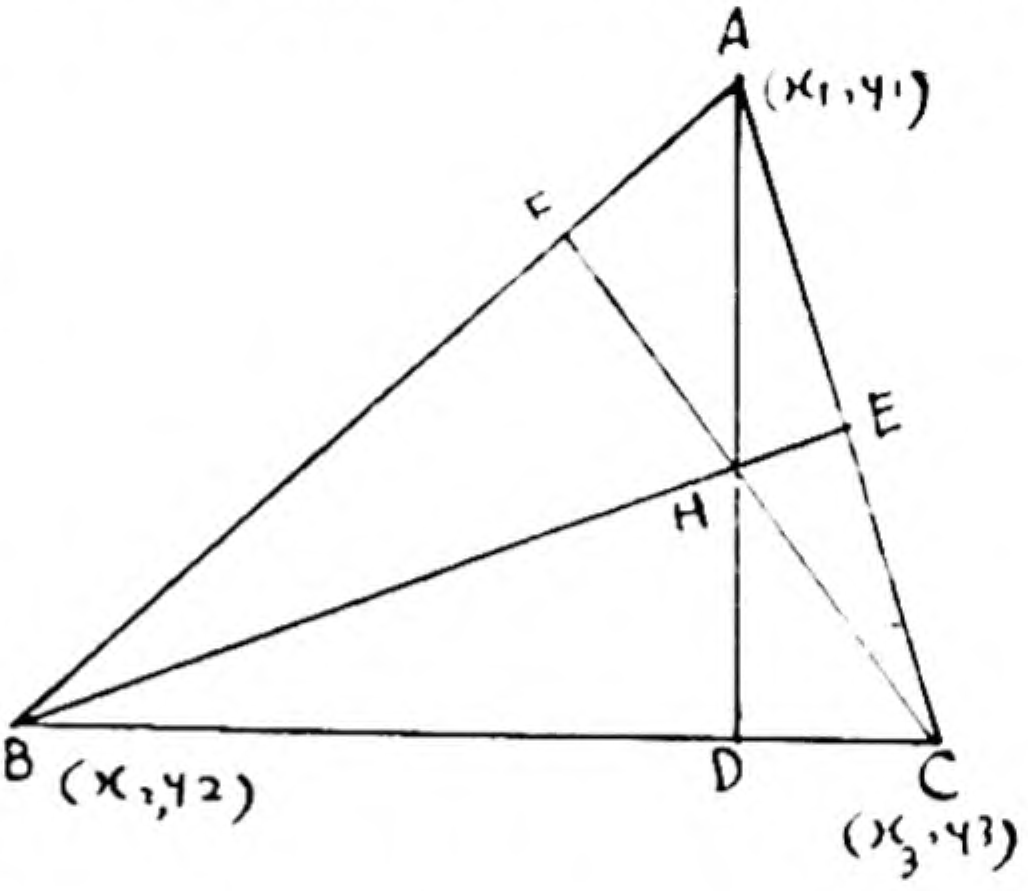
of $BC = \dfrac{y_3 - y_2}{x_3 - x_2}$

$\therefore$ the slope of AD, the perpendicular from A

on $BC = -\dfrac{x_3 - x_2}{y_3 - y_2}$

$$\cdots \left(m_1 = -\frac{1}{m_2}\right)$$

$\therefore$ the equation of AD is



$$y - y_1 = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1) \qquad \cdots \qquad [\because\ y - y_1 m = (x - x_1)]$$

or $x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0$ ...(1)

Similarly, the equations of the $\perp$s from B on CA, and from C on AB are :—

$$x(x_3 - x_1) + y(y_3 - y_1) - x_2(x_3 - x_1) - y_2(y_3 - y_1) = 0 \qquad \ldots(2)$$

and $x(x_1 - x_2) + y(y_1 - y_2) - x_3(x_1 - x_2) - y_3(y_1 - y_2) = 0$ ...(3)

Adding the L.H.S.'s of (1), (2), (3) vertically, we get

$$x(0) + y(0) + z(0) = 0$$

Hence the altitudes meet in a point.

Def : **Orthocentre.** *The point of concurrence of the altitudes of a triangle is called the orthocentre of the triangle.*

5.6. *To find the equation of any straight line through the intersection of the two straight lines*

$$a_1x + b_1y + c_1 = 0 \quad \textit{and} \quad a_2x + b_2y + c_2 = 0.$$

Let us consider the following equation :—

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0 \qquad \ldots(1)$$

Now (1) is an equation of first degree in x and y. Therefore, it represents a straight line. ...(Art. 3.2)

Again, let (x_1, y_1) be the point of intersection of the two straight lines represented by the equations

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0$$
$$\therefore \; a_1x_1+b_1y_1+c_1=0 \quad \text{and} \quad a_2x_1+b_2y_1+c_2=0$$

Now if (x_1, y_1) lies on the straight line represented by (1), then the L.H.S. of (1), *i.e.*,

$$a_1x_1+b_1y_1+c_1+k(a_2x_1+b_2y_1+c_2) \qquad ...(2)$$

must vanish identically. This is obviously so because the expressions

$$a_1x_1+b_1y_1+c_1 \quad \text{and} \quad a_2x_1+b_2y_1+c_2$$

vanish separately as (x_1, y_1) lies on

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0$$

Hence expression (2) becomes

$$(0)+k(0) \text{ i.e., } 0.$$

This shows that the point (x_1, y_1) also lies on the straight line represented by

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0$$

which, therefore, represents the straight line passing through the intersection of the two straight lines represented by

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0.$$

Note : This method is called the k-method, and, if applied, enables the student to save a lot of time in solving numerical questions. The value of k can be found from the second condition satisfied by the line.

Solved Examples

Ex. 1. Prove that the lines

$4x+7y-9=0$, $5x-8y+15=0$, and

$9x-y+6=0$ are concurrent.

Sol. Solving the first two equations simultaneously, we get

$$x=-\frac{33}{67} \text{ and } y=\frac{105}{67}$$

This obviously satisfies the third equation. Hence the three lines are concurrent and $\left(-\frac{33}{67}, \frac{105}{67}\right)$ is the point of concurrence.

Ex. 2. Find the area of the triangle the equations of whose sides are

$$y=m_1x+c_1,\ y=m_2x+c_2, \text{ and } x=0$$

Sol. The sides are :—

$$x=0 \qquad \ldots(1)$$

$$y=m_1x+c_1 \qquad \ldots(2)$$

$$y=m_2x+c_2 \qquad \ldots(3)$$

Intersection of (1) and (2) is $(0, c_1)$ —— A

Let us call this point A

Intersection of (2) and (3) is

$$\left(\frac{c_1-c_2}{m_2-m_1}, \frac{m_2c_1-m_1c_2}{m_2-m_1}\right) \text{ — } B$$

Call this point B.

Also, intersection of (1) and (3) is $(0, c_2)$ —— C.

Let this point be C.

Hence the area of the triangle ABC is

$$\frac{1}{2}\left[0\left(\frac{m_2c_1-m_1c_2}{m_2-m_1}-c_2\right)+\frac{c_1-c_2}{m_2-m_1}(c_2-c_1)+0\left(c_1-\frac{c_1-c_2}{m_2-m_1}\right)\right] \quad \ldots\text{(See Article}$$

$$=\frac{1}{2}\times\frac{c_1-c_2}{m_2-m_1}\times(c_2-c_1)=\tfrac{1}{2}\frac{(c_1-c_2)^2}{m_2-m_1}$$

Ex. 3. Find the equation of the straight line passing through the intersection of $x+2y+3=0$ and $3x+4y+7=0$ and perpendicular to the line $x-y+9=0$.

Sol. Solving the first two equations, we get

$$\frac{x}{14-12}=\frac{y}{9-7}=\frac{1}{4-6}$$

or $$\frac{x}{2}=\frac{y}{2}=\frac{1}{-2}$$

This gives $x=-1$, $y=-1$ as the point of intersection of these two lines.

Now, the equation of a st. line passing through $(-1, -1)$ is of the form

$$y+1=m(x+1) \qquad \ldots(1) \qquad [y-y_1=m(x-x_1)]$$

where m is its slope

If (1) is perpendicular on $x-y+9=0$

then $m=-1$ $\qquad \left(\because \; m_1=-\frac{1}{m_2}\right)$

$\therefore$ (1) becomes

$$y+1=-1(x+1)$$

or $$x+y+2=0$$

Second Method : *(To be preferred)*

The equation of the straight line through the intersection of $x+2y+3=0$ and $3x+4y+7=0$ is given by

$$x+2y+3+k(3x+4y+7)=0 \qquad \ldots(1) \qquad \text{(Art 5.6)}$$

Now (1) can be written as

$$x(1+3k)+2y(1+2k)+(3+7k)=0 \qquad \ldots(2)$$

The slope of (2) $=-\dfrac{1+3k}{2(1+2k)}$ (Article 3.15)

Further, the line represented by (2) is perpendicular to the line represented by

$x-y+9=0$ whose

slope is 1

$$\therefore \qquad -\frac{1+3k}{2(1+2k)}=-1 \qquad \left(\because \; m_1=-\frac{1}{m_2}\right)$$

This gives $k=-1$

Hence (2) becomes

$$-2x-2y-4=0$$

or $$x+y+2=0$$

Ex. 4. Find the equation of the straight line through (2, 3) and perpendicular on the line $y=3x+4$

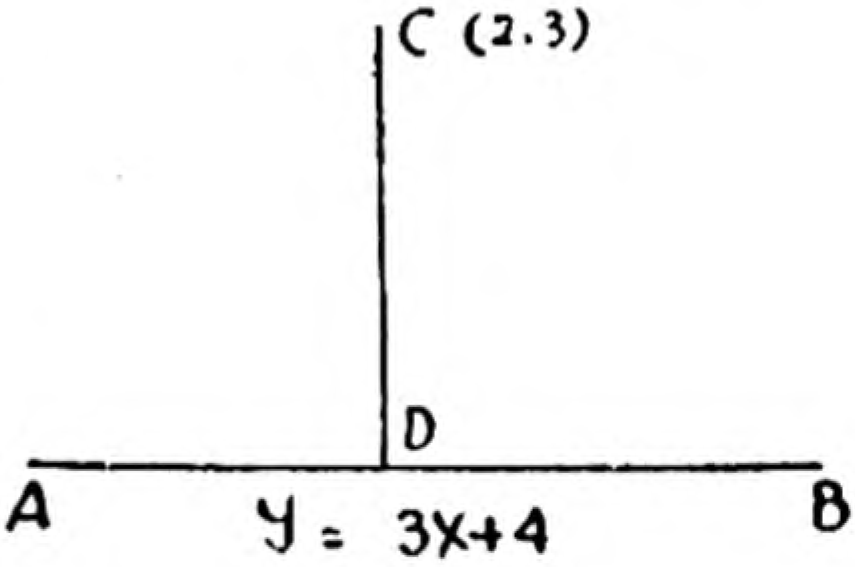
Find also the co-ordinates of the foot of the perpendicular.

Sol. The equation of a straight line passing through (2, 3) is of the form

$$y-3=m(x-2) \qquad \ldots(1)$$

$$[\because y-y_1=m(x-x_1)]$$

The slope of the line $y=3x+4$ is 3



$$\therefore \qquad m=-\frac{1}{3} \qquad \left(\because m_1=-\frac{1}{m_2}\right)$$

Hence (1) becomes

$$y-3=-\tfrac{1}{3}(x-2)$$

or $$x+3y-11=0 \qquad \ldots(2)$$

(*ii*) The co-ordinates of the foot of the perpendicular can be obtained by solving the equations $y=3x+4$ and $x+3y-11=0$ simultaneously. Thus we get

$$\frac{x}{11-12}=\frac{y}{4+33}=\frac{1}{9+1}$$

This gives $x=-\frac{1}{10}, y=\frac{37}{10}$

as the co-ordinates of the foot of the perpendicular.

Exercise 8

Find the co-ordinates of the points of intersections of the straight lines whose equations are :—

1. $2x-3y+5=0$ and $7x+4y=3$.

2. $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$.

3. $y=m_1x+\frac{a}{m_1}$ and $y=m_2x+\frac{a}{m_2}$.

4. $x \cos \phi_1+y \sin \phi_1=a$ and $x \cos \phi_2+y \sin \phi_2=a$.

5. $3x+y+8=0$ and $9x+3y+5=0$.

6. $t_1y=x+at_1^2$ and $t_2y=x+at_2^2$.

Show that the straight lines whose equations are given below are concurrent :—

7. $4x+7y-9=0$, $5x-8y+15=0$, $9x-y+6=0$.

8. $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, $y=x$.

9. $2x-3y=7$, $3x-4y=13$, $8x-11y=33$.

10. $15x-18y+1=0$, $12x+10y-3=0$, $6x+66y-11=0$.

Done

11. What must be the value of k in order that the three lines $3x+y-2=0$, $kx+2y-3=0$ and $2x-y-3=0$ may be concurrent ?

12. Find the co-ordinates of the ortho-centre of the triangle whose vertices are :

(*i*) $(0, 0)$, $(2, -1)$, and $(-1, 3)$.

(*ii*) $(1, 0)$, $(2, -4)$, and $(-5, --2)$.

13. Under what condition can the lines $y=m_1x+a_1$, $y=m_2x+a_2$, $y=m_3x+a_3$ be concurrent ?

Find the equations of the straight line

14. through $(3, 2)$ and the intersection of

$$2x+3y=1 \quad \text{and} \quad 3x-4y=6.$$

15. through $(2, -9)$ and the intersection of
$$2x+5y-8=0 \quad \text{and} \quad 3x-4y=35.$$

16. through the origin and the intersection of
$$x-y-4=0 \quad \text{and} \quad 7x+y+20=0.$$

17. through the intersection of the lines
$$x-2y-a=0 \quad \text{and} \quad x+3y-2a=0$$
and parallel to the line $3x+4y=0$.

18. through the intersection of the line
$$x+2y+3=0 \quad \text{and} \quad 3x+4y+7=0$$
and the perpendicular to the line $y-x=8$.

19. through the intersection of the lines
$$3x-4y+1=0 \quad \text{and} \quad 5x+y-1=0$$
and cutting off equal intercepts from the axes.

20. through the intersection of the lines
$$2x-3y=10 \quad \text{and} \quad x+2y=6$$
and the intersection of the lines
$$16x-10y=33 \quad \text{and} \quad 12x+14y+29=0.$$

21. find the area of the triangle formed by the lines whose equations are :—

(*i*) $y=x+1,\ x=2y-3,\ 3x-4y+7=0$
(*ii*) $3x+4y=2,\ x+2y=3,\ 3x+y=3$.

22. Show that the area of the triangle formed by the lines
$$y=m_r x+c_r, \quad \text{where} \quad r=1, 2, 3, \text{ is}$$
$$\frac{1}{2}\left[\frac{(c_1-c_2)^2}{m_1-m_2}+\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{(c_3-c_1)^2}{m_3-m_1}\right]$$

23. Show that the right bisectors of the sides of a triangle are concurrent.

24. Show that the medians of a triangle are concurrent.

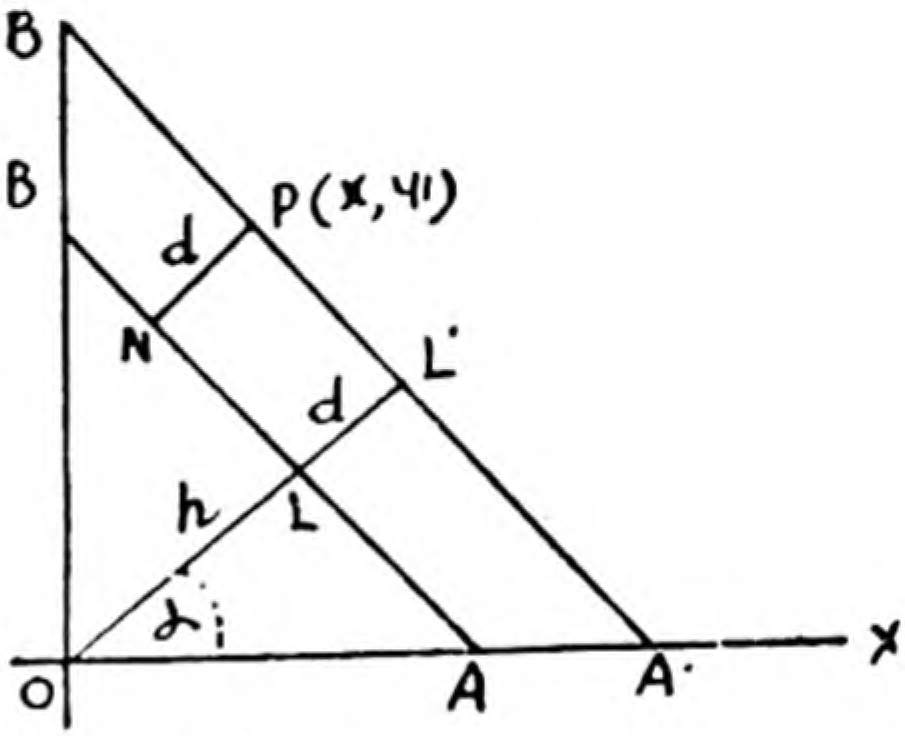
25. Find the orthocentre of the triangle whose vertices are $(0, 1)$, $(1, -2)$ and $(2, 3)$.

26. Find the co-ordinates of the circumcentre of the triangle whose vertices are $(-2, 2)$, $(2, -1)$, and $(4, 0)$.

LENGTH OF THE PERPENDICULAR FROM A GIVEN POINT ON A GIVEN LINE

6.1. When the equation of the line is given in the perpendicular form.

To find the perpendicular distance of the point (x_1, y_1) *from the line* $x \cos \alpha + y \sin \alpha = p$.



Let AB be the straight line represented by

$$x \cos \alpha + y \sin \alpha = p,$$

so that OL is the perpendicular from O on AB.

Now $OL = p$ and $\angle XOL = \alpha$.

Let P be the point (x_1, y_1), and PN $\perp$ on AB. Let $PN = d$. Through $P(x_1, y_1)$ draw A'B' parallel to AB meeting OL produced in L'.

Then $\quad OL' = OL + LL' = OL + PN = p + d$

and $\quad \angle XOL' = \alpha$

$\therefore$ the equation of AB is

$$x \cos \alpha + y \sin a = p + d \qquad \ldots(1)$$

Now (1) passes through $P(x_1, y_1)$

$\therefore \quad x_1 \cos \alpha + y_1 \sin \alpha = p + d$

Hence $\quad \mathbf{d = x_1 \cos \alpha + y_1 \sin \alpha - p}$

which gives the required perpendicular distance.

The length of the perpendicular from the point $P(x_1, y_1)$ *on the straight line* $x \cos \alpha + y \sin \alpha - p = 0$ *is, therefore, obtained by* *substituting the co-ordinates of the point in the left-hand side of the equation of the line.*

If, however, the equation of the line be $ax+by+c=0$, *it can be reduced to the perpendicular form as*

$$\frac{a}{\sqrt{a^2+b^2}}\,x+\frac{b}{\sqrt{a^2+b^2}}\,y+\frac{c}{\sqrt{a^2+b^2}}=0 \quad \text{(Article 3.13)}$$

$\therefore$ *the length of the perpendicular from* (x_1, y_1) *on the line is*

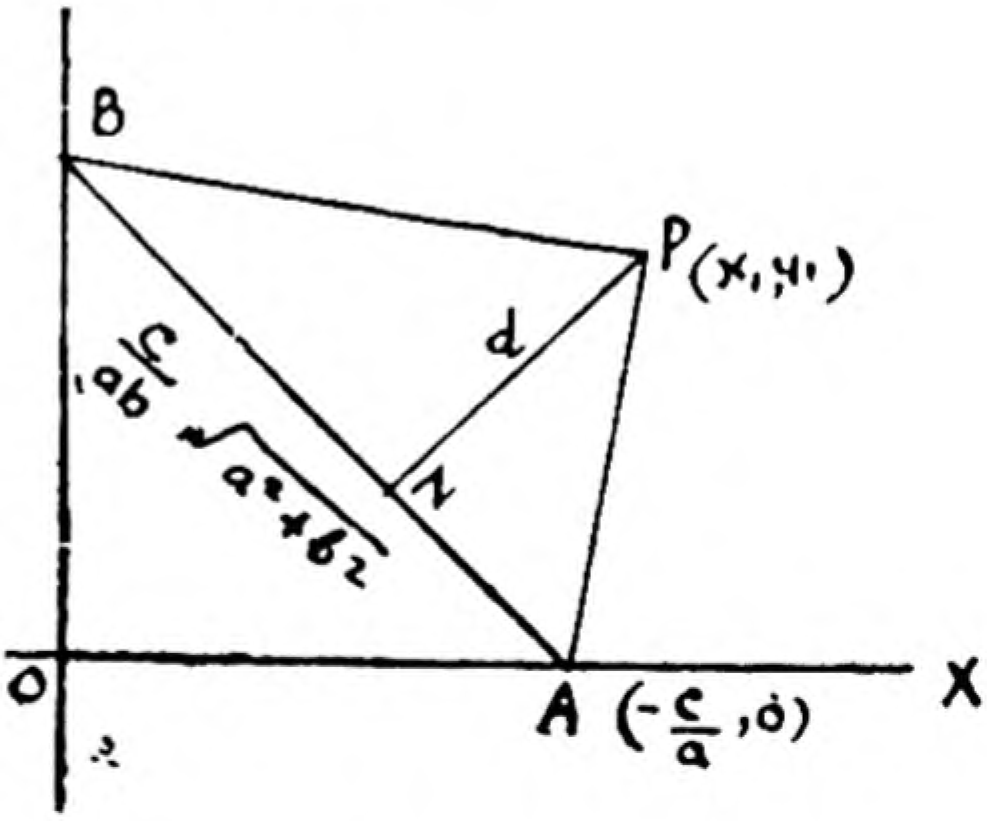
$$\frac{a}{\sqrt{a^2+b^2}}\,x_1+\frac{b}{\sqrt{a^2+b^2}}\,y_1+\frac{c}{\sqrt{a^2+b^2}}$$

i.e.,

$$\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}.$$

Hence the length of the perpendicular from (x_1, y_1) *on the line* $ax+by+c=0$ *is obtained by substituting the co-ordinates of the point in* $ax+by+c$ *and dividing it by* $\sqrt{a^2+b^2}$.

Second Method : (*Independent method*)



Let $ax+by+c=0$ be the equation of the line, and PN the perpendicular drawn from $P(x_1, y_1)$ on the line.

Let $PN=d$.

The line $ax+by+c=0$ will meet the x-axis in A whose co-ordinates will be $\left(-\frac{c}{a}, 0\right)$.

Similarly, it will meet the y-axis in B whose co-ordinates will be $\left(0, -\frac{c}{b}\right)$

Join PA and PB

$$\text{Now } AB=\sqrt{\left(0+\frac{c}{a}\right)^2+\left(-\frac{c}{b}-0\right)^2}=\frac{c}{ab}\sqrt{a^2+b^2}$$

$$\therefore \quad \triangle PAB = \tfrac{1}{2}.AB.PN = \tfrac{1}{2}.\frac{c}{ab}\sqrt{a^2+b^2}.d \qquad \ldots(1)$$

But the area of the $\triangle$ PAB in terms of its vertices

$$= \tfrac{1}{2}\left[x_1\left(-\frac{c}{b}-0\right)+0\left(0-y_1\right)-\frac{c}{a}\left(y_1+\frac{c}{b}\right)\right]$$

(Article 1.14)

$$= \tfrac{1}{2}\left[-\frac{cx_1}{b}-\frac{cy_1}{a}-\frac{c^2}{ab}\right]$$

$$= -\frac{c}{2ab}\left(ax_1+by_1+c\right)$$

$$= \frac{c}{2ab}\left(ax_1+by_1+c\right) \ldots(2) \qquad \ldots(\text{rejecting} -\text{ve sign})$$

Now $(1)=(2)$

$$\therefore \quad \frac{c}{2ab}\sqrt{a^2+b^2}\,.\,d = \frac{c}{2ab}\left(ax_1+by_1+c\right)$$

Hence $d = \dfrac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}$

Third Method : *(To be preferred)*

The slope of the line $ax+by+c=0$ is $-\dfrac{a}{b}$.

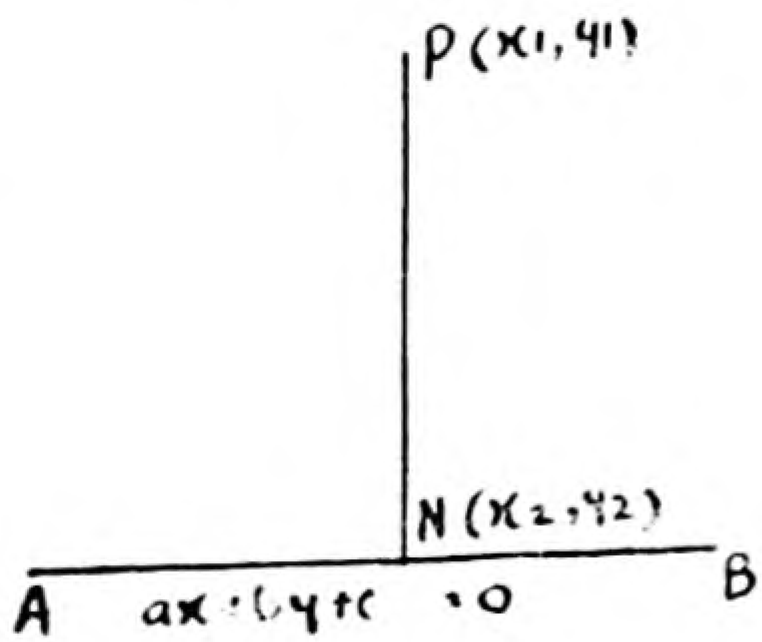
$\therefore$ the slope of the line through $P(x_1, y_1)$ perpendicular to $ax+by+c=0$ is $\dfrac{b}{a}$

Let PN be the $\perp$ on AB

$\therefore$ the equation of PN is

$$y-y_1=\frac{b}{a}(x-x_1)\ldots\ldots$$

or $b(x-x_1)-a(y-y_1)=0 \quad \ldots(1)$



If N(x_2, y_2) be the foot of the perpendicular PN, then, as (x_2, y_2) is common to both (1) and $ax+by+c=0$, we have

$$b(x_2-x_1)-a(y_2-y_1)=0 \qquad \ldots(2)$$

and $ax_2+by_2+c=0$, which we may write as

$$a(x_2-x_1)+b(y_2-y_1)=-(ax_1+by_1+c) \qquad \ldots(3)$$

Now $\text{PN}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad \ldots(4)$

$\therefore$ from (2) and (3) by squaring and adding, we have

$$(a^2+b^2)\{(x_2-x_1)^2+(y_2-y_1)^2\}=(ax_1+by_1+c)^2$$

or

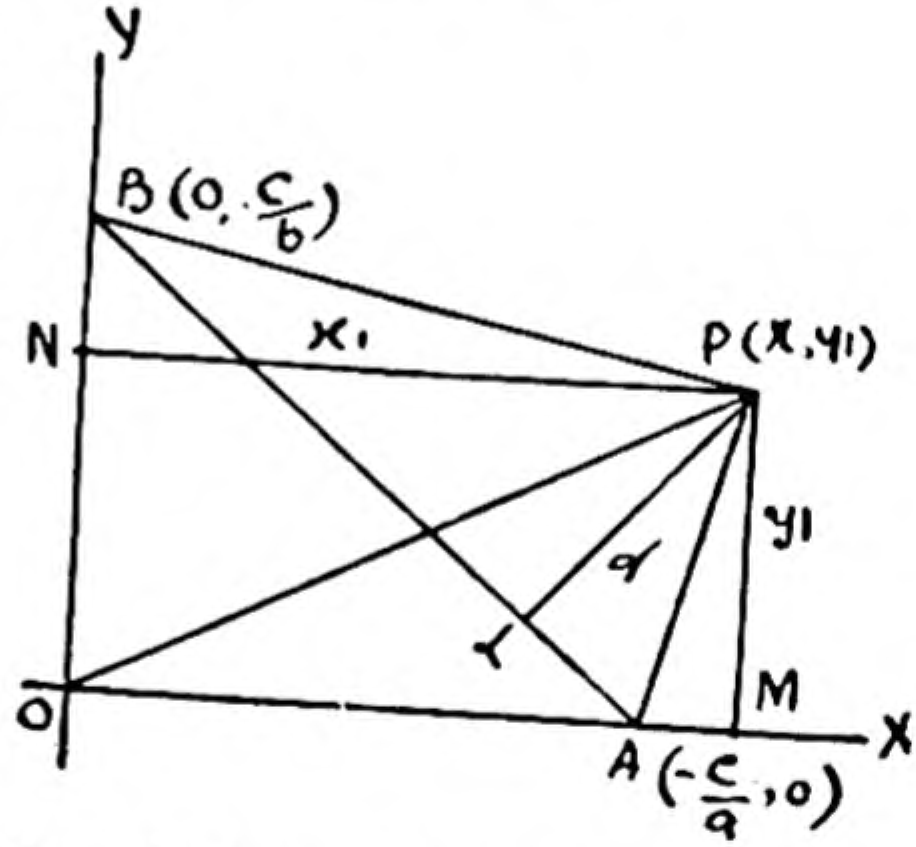
$$(a^2+b^2)\ \text{PN}^2=(ax_1+by_1+c)^2$$

[from (4)]

$$\therefore \qquad \mathbf{PN=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}}$$

Fourth Method :

Let AB be the line whose equation is $ax+by+c=0$



Let it cut OX and OY in A and B, so that the co-ordinates of these points are $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$ respectively.

Then $\text{AB}=\frac{c}{ab}\sqrt{a^2+b^2}$.

Join AP, BP, OP where P(x_1, y_1) is the point from PL$=d$ is drawn as perpendicular on AB. Draw PM, PN$\perp$s on OX and OY respectively. Then PM$=y_1$ and PN$=x_1$.

Now $\qquad \triangle\text{APB}+\triangle\text{OAB}=\triangle\text{OAP}+\triangle\text{OPB}$

or $\qquad \frac{1}{2}\,\text{AB.PL}+\frac{1}{2}\,\text{OA.OB}=\frac{1}{2}.\text{OA.PM}+\frac{1}{2}.\text{OB.PN}$

or $\frac{c}{ab}\sqrt{a^2+b^2}\times d+\left(-\frac{c}{a}\right)\left(-\frac{c}{b}\right)=\left(-\frac{c}{a}\right)\times y_1+\left(-\frac{c}{b}\right)\times x_1$

or $$\frac{c}{ab}\sqrt{a^2+b^2}.d=-\left(\frac{cy_1}{a}+\frac{cx_1}{b}+\frac{c^2}{ab}\right)$$

or $$\sqrt{a^2+b^2}.d=-(ax_1+by_1+c)$$

Hence $$\mathbf{d}=\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}$$

(rejecting the $-$ve sign)

6.2. *To prove that the two points* (x_1, y_1) *and* (x_2, y_2) *will lie on the same or opposite sides of a given line* $ax+by+c=0$, *according as* ax_1+by_1+c *and* ax_2+by_2+c *have the same or opposite signs.*

Let $l : m$ be the ratio in which the line $ax+by+c=0$ divides the line joining (x_1, y_1) and (x_2, y_2). Therefore, the point $\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}\right)$ lies on $ax+by+c=0$

$$i.e., \quad a.\frac{lx_2+mx_1}{l+m}+b.\frac{ly_1+my_1}{l+m}+c=0$$

or $$\frac{l}{m}=-\frac{ax_1+by_1+c}{ax_2+by_2+c} \qquad ...(1)$$

Now if the point $\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}\right)$ be between the two given points (x_1, y_1) and (x_2, y_2) *i.e.*, if the points (x_1, y_1) and (x_2, y_2) lie on the opposite sides of $ax+by+c=0$, then $\frac{l}{m}$ is positive.

In this case, by (1), the quantities ax_1+by_1+c and ax_2+by_2+c must have opposite signs.

The two points (x_1, y_1) *and* (x_2, y_2) *are, therefore, on the opposite sides of the line* $ax+by+c=0$ *if the quantities* ax_1+by_1+c *and* ax_2+by_2+c *have different signs.*

Again, if the point $\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}\right)$ does not lie anywhere in between (x_1, y_1) and (x_2, y_2) *i.e.*, if (x_1, y_1) and (x_2, y_2) lie on the same side of the line $ax+by+c=0$, then the point $\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}\right)$ is the point of *external* division of the points (x_1, y_1) and (x_2, y_2), so that $\frac{l}{m}$ is negative.

In that case, by (1) the two quantities ax_1+by_1+c and ax_2+by_2+c must have the same sign.

The two points (x_1, y_1) *and* (x_2, y_2), *therefore, lie on the same side of the line* $ax+by+c=0$ *if the quantities* ax_1+by_1+c *and* ax_2+by_2+c *have the same sign.*

6.3. Bisectors of angles between the straight lines

To find the equations of the bisectors of the angles between the straight lines $a_1x+b_1y+c=0$ *and* $a_2x+b_2y+c_2=0$

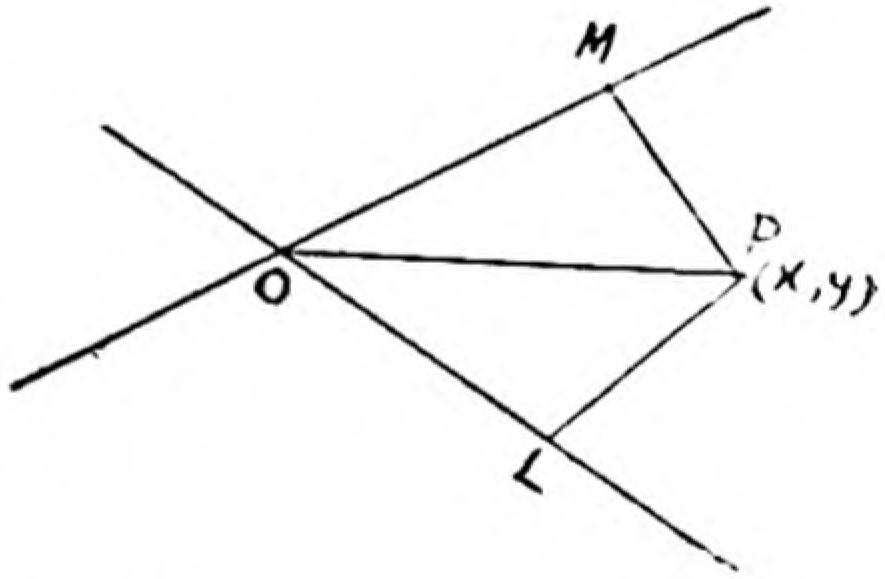
The equations of the lines are

$a_1x+b_1y+c_1=0$ ...(1)

$a_2x+b_2y+c_2=0$ ...(2)

Let these two lines intersect at O. Let P(x, y) be any point on either of the bisectors. Then the length of the ⊥ from (x, y) on (1)

= the length of the ⊥ from (x, y) on (2)



i.e. $$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1}^2+\mathbf{b_1}^2}}=\pm\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2}^2+\mathbf{b_2}^2}}$$

which give the required equations of the bisectors.

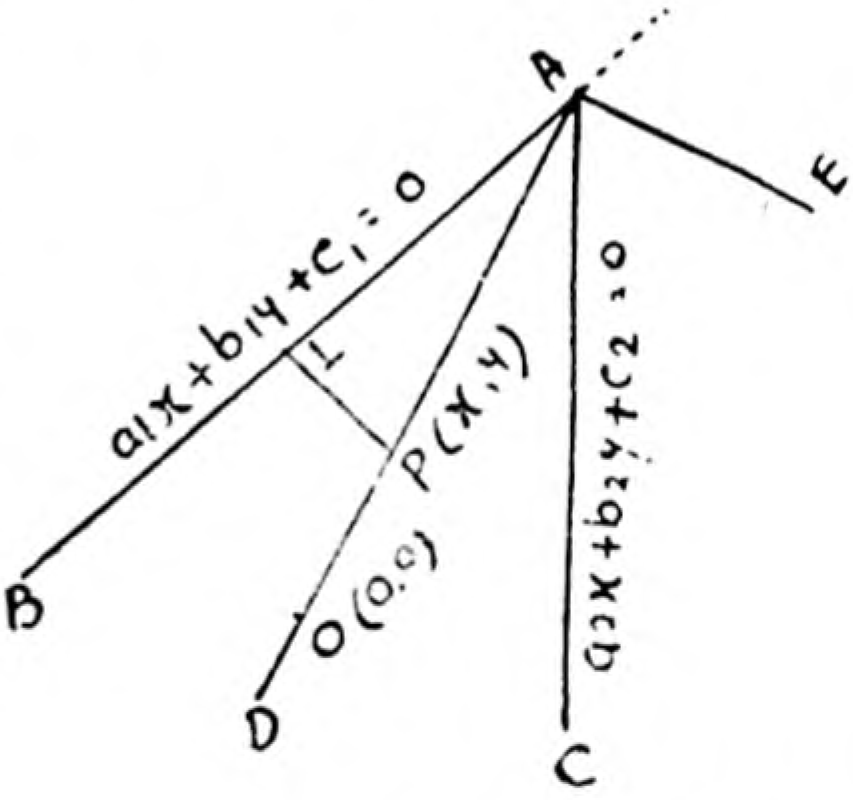
Hence, we lay down the following working rule for finding the equations of the bisectors of the angles between two given lines.

(*i*) *Write the equations of the lines so that their R.H.S.'s are zero,* (*ii*) *take the L.H S. of the first equation and divide it by* $\sqrt{(\textit{coeff}.x)^2+(\textit{coeff}.y)^2}$

(*iii*) *take the L H.S. of the second equation and divide it by* $\sqrt{(\textit{coeff}\ x)^2+(\textit{coeff}.y)^2}$

(*iv*) *Equate the results obtained in* (*ii*) *and* (*iii*) *taking the double sign* ($\pm$) *on the R.H.S.*

6.4. *To find the equation of the angle between the lines* $a_1x+b_1y+c_1=0$ *and* $a_2x+b_2y+c_2=0$ *in which the origin lies,* c_1, c_2 *being both positive.*



Let AB and AC be the lines

$$a_1x+b_1y+c_1=0 \qquad \ldots(1)$$
$$a_2x+b_2y+c_2=0 \qquad \ldots(2)$$

Let P(x, y) be any point on AD, the bisector of $\angle$BAC, in which the origin lies.

Now P(x, y) and O(0, 0) are on the same side of
$$a_1x+b_1y+c_1=0$$

$\therefore$ the quantities $a_1x+b_1y+c_1$ and $a_1(0)+b_1(0)+c_1$, *i.e.*, $a_1x+b_1y+c_1$ and c_1 must be of the same sign (Article 6.2)

But c_1 is given to be positive

$\therefore$ $a_1x+b_1y+c_1$ is also positive

$\therefore$ length of the $\perp$ from P(x, y) on (1)

$$=\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$$

For similar reasons, the length of the $\perp$ from P(x, y) on (2)

$$=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Hence $$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1^2+b_1^2}}}=\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2^2+b_2^2}}}$$

is the equation of the bisector of $\angle BAC$ in which the origin lies.

Similarly, *the equation of the bisector of $\angle BAC$ in which the origin does not lie is*

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1^2+b_1^2}}}=-\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2^2+b_2^2}}}$$

6.5. *To prove that the bisectors of the angles of a triangle are concurrent.*

Let us choose the origin as the point lying within by triangle whose sides are

$$a_1x+b_1y+c_1=0 \quad \ldots(i)$$

$$a_2x+b_2y+c_2=0 \quad \ldots(ii)$$

$$a_3x+b_3y+c_3=0 \quad \ldots(iii)$$

Now the internal bisectors of the angles are the bisectors of the angles which contain the origin, since the origin lies within the triangle ; therefore, their equations are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

or $$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}-\frac{a_2x+b_3y+c_2}{\sqrt{a_2^2+{b^2}_2}}=0 \quad \ldots(iv)$$

$$\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}-\frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}}=0 \quad \ldots(v)$$

and $$\frac{a_3x+b_2y+c_3}{\sqrt{a_3^2+b_3^2}}-\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=0 \quad \ldots(vi)$$

Evidently, the sum of the L.H.S.'s of (iv), (v) and (vi) is zero. *Hence by Article 5.5, the bisectors meet in a point.*

An Important Note :

In order to find the equation of the internal bisector of an angle of a triangle, write the equations of the surrounding sides, and *change the signs throughout, if necessary, so that on substituting the co-ordinates of the opposite vertex in the R.H.S. of each such equation (R.H.S. being zero) is positive.*

Solved Examples

Ex. 1. Find (*i*) the equation of the perpendicular (*ii*) the length of the perpendicular, and (*iii*) the co-ordinates of the foot of the perpendicular from the point (3, 4) to the line $8x+15y+1=0$

Sol. The given line is

(*i*) $$8x+15y+1=0 \qquad \ldots(1)$$

$\therefore$ its slope $=-\frac{8}{15}$ (Article 3.14)

$\therefore$ the slope of the line $\perp$ to (1) is $\frac{15}{8}$......

$$\left(\because \ m_1=-\frac{1}{m_2}\right)$$

Hence the equation of the perpendicular through (3, 4) is

$$y-4=\tfrac{15}{8}(x-3) \qquad [y-y_1=m(x-x_1)]$$

or $$15x-8y-13=0 \qquad \ldots(2)$$

(*ii*) the length of the $\perp$ from (3, 4) on (1) is given by

$$d=\frac{8.3+15.4+1}{\sqrt{8^2+15^2}}=\frac{85}{17}=5 \qquad \text{(Article 6.1)}$$

(*iii*) To get the co-ordinates of the foot of the perpendicular we have to solve (1) and (2)

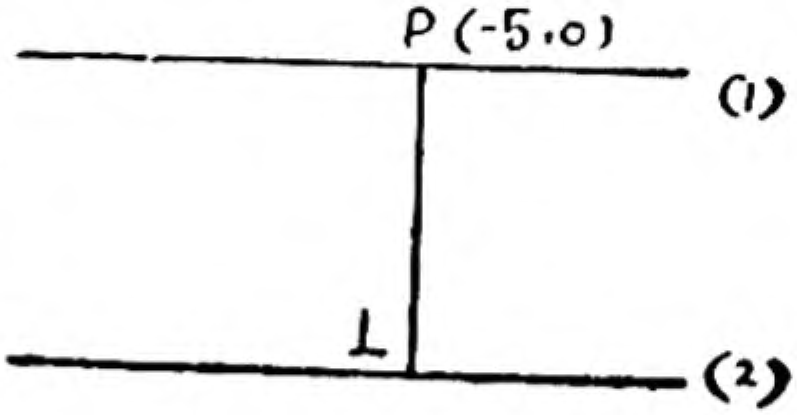
Thus $$\frac{x}{-195+8}=\frac{y}{15+104}=\frac{1}{-64-225}$$

which gives $\left(\frac{11}{17}, -\frac{7}{17}\right)$ as the co-ordinates of the foot of the $\perp$.

Ex. 2. *Find the distance between the lines* $3x+4y+15=0$ *and* $3x+4y-9=0$.

Sol. The two lines are

$$3x+4y+15=0 \qquad \ldots(1)$$
$$3x+4y-9=0 \qquad \ldots(2)$$



Evidently, these are parallel to each other. Take P any point on (1) so that the ordinate of P is zero.

∴ the co-ordinates of P are $(-5, 0)$

Draw PL ⊥ on (2). Let $PL=d$, then

$$d=\frac{3\times-5+4\times0-9}{\sqrt{3^2+4^2}}=\frac{-15-9}{5}=\frac{24}{5}$$

(After rejecting the $-ve$ sign)

Alternative Method : (*to be preferred*)

Let OL and OM be ⊥s on (1) and (2) respectively. Then the required perpendicular distance between (1) and (2) is LOM

Now
$$OL=\frac{3\times0+5\times0+15}{\sqrt{3^2+4^2}}=3 \qquad \ldots(1)$$
$$OM=\frac{3\times0+5\times0-9}{\sqrt{3^2+4^2}}=\frac{9}{5} \qquad \ldots(2)$$

(rejecting $-ve$ sign)

Adding (1) and (2), we get

$$OL+OM=LOM=\frac{24}{5}$$

Ex. 3. Find the bisector of the obtuse angle between the lines

$$12x-5y-5=0 \qquad \ldots(i)$$
$$3x+4y-6=0 \qquad \ldots(ii)$$

Sol. The bisectors of the angles between (*i*) and (*ii*) are given by

$$\frac{12x-5y-5}{13}=\pm\frac{3x+4y-6}{5} \qquad \text{(Article 6.3)}$$

This gives

$$21x-77y+53=0 \qquad \ldots(iii)$$

and $$99x+27y-103=0 \qquad \ldots(iv)$$

as the bisectors.

Now any point on (*i*) is (0, −1)

∴ Length of the ⊥ from (0, −1) on (*iii*)

$$=\frac{21\times 0-77\times -1+53}{\sqrt{(21)^2+(77)^2}}=\frac{\sqrt{130}}{7}$$

Also the length of the ⊥ from (0, −1) on (*iv*)

$$=\frac{99\times 0+27\times -1-103}{\sqrt{(99)^2+(27)^2}}=-\frac{\sqrt{130}}{9}$$

The first perpendicular is longer than the second one in *magnitude*. Hence (*iii*) gives the equation of the bisector of the obtuse angle.

This can be shown even as under :—

If (*iii*) is the bisector of the obtuse angle, then the tangent of the angle between (*iii*) and (*i*) [or between (*iii*) and (*ii*)] must be greater than unity

Thus $$\tan\theta=\frac{21\times -5+12\times -77}{21\times 12+-5\times -77}=-\frac{21}{19}$$

which is greater than 1 *numerically*

Hence (*iii*) is the bisector of the obtuse angle.

Exercise 9

1. Prove that the origin and the point (−1, −2) are on the opposite sides of $2x+3y+5=0$.

2. Are the points (−2, 3) and (−2, 4) on the same or the opposite sides of the line $4x+5y+10=0$?

3. Find the distance of the point $(2\sqrt{3}, 8)$ from the line

$$x \cos\frac{\pi}{6} + y \sin\frac{\pi}{6} = 5.$$

4. Find the length of the perpendicular drawn from

   (*i*) the point $(2, -1)$ upon the straight line
   $$3x - 4y = 5$$

   (*ii*) the point (b, a) upon the straight line
   $$\frac{x}{a} - \frac{y}{b} = 1$$

   (*iii*) the origin upon the line $h(x+h) + k(y+k) = 0$

   (*iv*) the point $(-2, 3)$ upon the line $x - y = 5$.

   (*v*) the point (b, a) upon the line $a(x-a) = b(b-y)$.

5. Find the length of the perpendicular and the co-ordinates of the foot of the perpendicular from the point $(3, 4)$ to the line
$$8x + 15y + 1 = 0$$

6. If p is the perpendicular distance of the origin from a line whose intercepts on the axes are 'a' and 'b'

show that
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$
(*K.U. 1952*)

7. If p and p' be the perpendiculars from the origin upon the straight lines whose equations are
$$x \sec\theta + y \operatorname{cosec}\theta = a$$
and
$$x \cos\theta - y \sin\theta = a \cos 2\theta$$
prove that
$$4p^2 + p'^2 = a^2.$$

8. Find the lengths of the altitudes of a triangle whose angular points are $(0, 0)$, $(1, -1)$, and $(3, 2)$.

9. Show that the product of the perpendiculars drawn from the two points $(\pm\sqrt{a^2 - b^2}, 0)$ upon the line
$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$
is b^2.

10. Find the perpendicular distance between the lines :—

(*i*) $3x+4y+5=0$ and $3x+4y+17=0$.

(*ii*) $y=2x-4$ and $y=2x+3$.

(*iii*) $ax+by+c=0$ and $a'x+b'y+c'=0$.

(*iv*) $y=mx+c$ and $y=mx+d$.

11. Find the bisector of the acute angle between the lines

$$3x+4y=11 \quad \text{and} \quad 12x-5y=2.$$

12. Find the bisector of the acute angle between the lines

$$5x=12y+24 \quad \text{and} \quad 12x=5y+10.$$

13. Find the equation of the line passing through the point (2, 9) and the intersection of the lines

$$3x+4y-7=0 \quad \text{and} \quad 5x-12y+7=0.$$

Show that it bisects the obtuse angle between the given lines, and find also the bisector of the acute angle.

14. Find the centre and the radius of the circle which is inscribed in the triangle formed by the lines

$$y=0,\ 12x-5y=0,\ 3x+4y-7=0$$

15. Show that the origin is within the triangle whose angular points are

$$(1, 2),\ (-2, 3), \quad \text{and} \quad (-1, -4).$$

$-3/4$

CHAPTER VII

JOINT EQUATION OF TWO OR MORE STRAIGHT LINES

7.1. We have seen in Article 3.2 that the general equation of the first degree is x and y of the type $ax+by+c=0$ represents a straight line. Let us now consider the product of the two equations

$$(ax+by+c)(a'x+b'y+c')=0 \qquad \dots(1)$$

to see what it represents.

$$ax+by+c \quad \text{and} \quad a'x+b'y+c'$$

are the two factors on the L.H.S. of equation (1). This equation is, therefore, satisfied if either of these factors is zero. Thus equation (1) represents the pair of straight lines

$$ax+by+c=0 \quad \text{and} \quad a'x+b'y+c'=0.$$

Again, on multiplying the two factors on the left-hand side of (1), we will get an equation of the form

$$\text{A}x^2+2\,\text{H}xy+\text{B}y^2+2\,\text{G}x+2\text{F}y+\text{C}=0 \qquad \dots(2)$$

where the values of A. B, H, C, G, F can be easily determined in terms of $a, b. c, a', b'$, and c'. This is the *most* general equation of the second degree and will represent a repair of straight lines provided that its L.H.S. can be factorized into two linear factors in x and y.

7.2. Homogeneous equation of the nth degree.

An equation in x and y of the type

$$a_0x^n+a_1x^{n-1}y+a_2y^{n-2}y^2+\dots\dots+a_ny^n=0$$

in which the sum of the indices of x and y of each term is equal to n is called the homogeneous equation of the nth degree in x and y.

Thus $ax^2+2hxy+by^2=0$ is a homogeneous equation of the second degree in x and y, whereas

$$ax^3+bx^2y+cxy^2+dy^3=0$$

is a homogeneous equation of the 3rd degree.

7.3 To show that the homogeneous equation of the second degree

$$\mathbf{ax^2+2hxy+by^2=0}$$

represents a pair of straight lines passing through the origin.

Solving the equation for x, we get

$$x=\frac{-h\pm\sqrt{h^2-ab}}{a}y$$

This gives $\quad ax+(h+\sqrt{h^2-ab})y=0 \qquad \ldots(i)$

and $\quad ax+(h-\sqrt{h^2-ab})y=0 \qquad \ldots(ii)$

Each of the equations (i) and (ii) being of *first degree* in x and y *represents* a straight line through the origin.

Hence $\mathbf{ax^2+2hxy+by^2=0}$ represents two straight lines passing through the origin.

7.4. Angle between the two straight lines represented by $\mathbf{ax^2+2hxy+by^2=0}$ ...(1)

Let the two lines represented by

$$ax^2+2hxy+by^2=0$$

be $\quad y=m_1x \quad$ and $\quad y=m_2x$

Then $\quad (y-m_1x)\,(y-m_2x)=0$

i.e., $\quad m_1m_2x^2-(m_1+m_2)xy+y^2=0$

is the same as (1)

comparing the coefficients, we get

$$\frac{m_1m_2}{a}=-\frac{m_1+m_2}{2h}=\frac{1}{b}$$

which gives

$$m_1 m_2 = \frac{a}{b}$$

and $$m_1 + m_2 = -\frac{2h}{b}$$

Thus, if θ be the angle between the straight lines represented by (1) we have

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{4h^2 - 4ab}}{a+b}$$

Hence $$\theta = \tan^{-1}\left\{\frac{2\sqrt{h^2-ab}}{a+b}\right\}$$

Cor. If $a+b=0$, *the two lines are perpendicular,* whereas these are coincident if $h^2=ab$. **Thus the two lines will be perpendicular if the sum of coefficients of x^2 and y^2 is zero.**

7.5. To show that a homogeneous equation of the nth degree in x and y represents n straight lines passing through the origin.

Let $ay^n + by^{n-1}x + cy^{n-2}x^2 + \ldots\ldots + kx^n = 0$ ...(1)
be the given homogeneous equation of the *nth* degree.

Dividing (1) throughout by x^n, we get

$$a\left(\frac{y}{x}\right)^n + b\left(\frac{y}{x}\right)^{n-1} + c\left(\frac{y}{x}\right)^{n-2} + \ldots\ldots + k = 0 \quad \ldots(2)$$

This is an equation of the *nth* degree in $\frac{y}{x}$, and will have *n* roots as such.

Let $m_1, m_2, m_3, \ldots\ldots m_n$ be these roots

Therefore, (2) can be written as

$$a\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\left(\frac{y}{x} - m_n\right) = 0$$

or $$a(y-m_1x)\,(y-m_2x)\ldots\ldots(y-m_nx)=0$$

Since $a\neq 0$, we have

$$y-m_1x=0,\ y-m_2x=0,\ldots\ldots y-m_nx=0$$

which are the equations of n straight lines passing through $(0, 0)$.

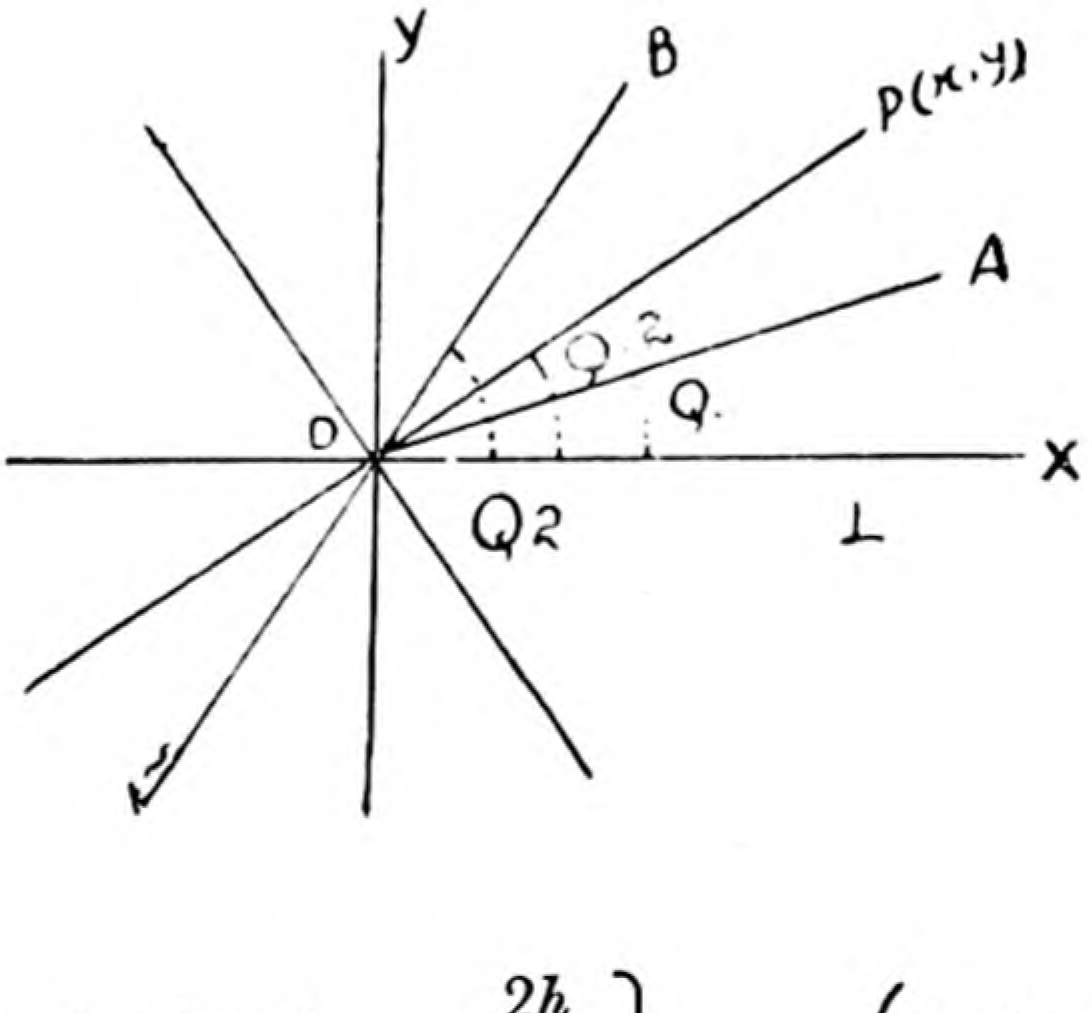
Hence (1) represents n straight lines passing through the origin.

Note : The n straight lines need not be *necessarily real* and *distinct.*

7.6. To find the equation of the straight lines bisecting the angles between the lines

$$\mathbf{ax^2+2hxy+by^2=0}$$

Let θ_1 and θ_2 be the angles at which the two given lines represented by $ax^2+2hxy+by^2=0$ are inclined to the x-axis.



$$\therefore\quad \left.\begin{aligned} \tan\theta_1+\tan\theta_2&=-\frac{2h}{b} \\ \text{and}\quad \tan\theta_1.\tan\theta_2&=\frac{a}{b} \end{aligned}\right\}\ \ldots(1) \qquad \begin{aligned} &\left(\because\ m_1+m_2=-\frac{2h}{b}\right) \\ &\left(\because\ m_1m_2=\frac{a}{b}\right) \end{aligned}$$

Let θ be the angle which a bisector makes with the x-axis, then

$$\theta=\frac{\theta_1+\theta_2}{2} \text{ or } \theta=\frac{\theta_1+\theta_2}{2}+\frac{\pi}{2}$$

and in either case

$$\tan 2\theta=\tan(\theta_1+\theta_2) \qquad \ldots\ldots[\because \tan(\pi+\theta)=\tan\theta]$$

or

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} \qquad \ldots(2)$$

Now, let P(x, y) be any point on the bisector, then

$$\tan\theta=\frac{\text{PL}}{\text{OL}}=\frac{y}{x} \qquad \ldots(3)$$

Hence from (1), (2) and (3), we get

$$\frac{2\frac{y}{x}}{1-\frac{y^2}{x^2}}=\frac{-\frac{2h}{b}}{1-\frac{a}{b}}$$

or

$$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}}=\frac{\mathbf{xy}}{\mathbf{h}}$$

which gives the joint equation of the two bisectors.

Solved Examples

Ex. 1. Find the angle between the lines

$$x^2+2xy\sec\theta+y^2=0 \qquad (K.\ U.)$$

Sol. Comparing this equation with

$$ax^2+2hxy+by^2=0,$$

we have $a=1$, $h=\sec\theta$, $b=1$.

Let ϕ be the angle between the lines.

$$\therefore \quad \tan\phi=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\sec^2\theta-1}}{2}=\tan\theta$$

$$\therefore \quad \phi=\theta$$

Ex. 2. Prove that the equation to the pair of straight lines at right angles to the pair given by

$$ax^2+2hxy+by^2=0 \text{ is}$$
$$bx^2-2hxy+ay^2=0$$

Sol. The equation $ax^2+2hxy+by^2=0$ can be put as

$$b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0 \qquad \text{...(1)}$$

Now the given equation represents two st. lines passing through the origin. The equations of these straight lines taken separately are of the form

$$y=m_1x \quad \text{and} \quad y=m_2x$$

$\therefore$ in both the cases their slopes are $\frac{y}{x}$ $\left(\because \quad m_1=m_2=\frac{y}{x}\right)$

Hence the equation of the st. lines ⊥ to the st. lines represented by (1) will be obtained by substituting

$$\left(-\frac{x}{y}\right) \text{ for } \left(\frac{y}{x}\right) \text{ in (1)}$$

The equation of the st. lines will, therefore, be

$$b\left(-\frac{x}{y}\right)^2+2h\left(-\frac{x}{y}\right)+a=0$$

or
$$ay^2-2hxy+bx^2=0$$

Ex. 3. Show that the condition that the two of the lines represented by $ay^3+bxy^2+cx^2y+dx^3=0$ may be at right angles is $a(a+c)+d(b+d)=0$ (*P. U. 1953*)

Sol. The given equation can be written as

$$a\left(\frac{y}{x}\right)^3+b\left(\frac{y}{x}\right)^2+c\left(\frac{y}{x}\right)+d=0 \qquad \text{...(1)}$$

Putting $\frac{y}{x}=m$ in (1) we get

$$am^3+bm^2+cm+d=0$$

Let m_1, m_2, m_3 be the roots of (2)

$$\therefore \qquad m_1m_2\,m_3=-\frac{d}{a} \qquad \ldots(3)$$

But two of the lines represented by (1) are at right ∠s. Let their slope be m_1 and m_2

$$\therefore \qquad m_1m_3=-1$$

$$\therefore \qquad \text{(2) gives} \quad m_3=\frac{d}{a}$$

Now m_3 is a root of (2)

$$\therefore \qquad a\left(\frac{d}{a}\right)^3+b\left(\frac{d}{a}\right)^2+c\left(\frac{d}{a}\right)+d=0$$

or $$d^2+bd+ac+a^2=0$$

or $$a(a+c)+d(b+d)=0.$$

Ex. 4. Prove that the product of the perpendicular from the point (x_1, y_1) on the lines $ax^2-2hxy+by^2=0$ is equal to

$$\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}$$

Let $y-m_1x=0$ and $y-m_2x=0$ be the straight lines represented by $ah^2-2hxy+by^2=0$ so that

$$m_1+m_2=-\frac{2h}{b} \text{ and } m_1\,m_2=\frac{a}{b}$$

Now the product of the perpendiculars from (x_1, y_1) on the two lines $y-m_1x=0$ and $y-m_2x=0$ is given by

$$p_1\,p_2=\frac{y_1-m_1x_1}{\sqrt{1+m_1^2}}\cdot\frac{y_1-m_2x_1}{\sqrt{1+m_2^2}}$$

where p_1, p_2 are the ⊥s on these lines

$$=\frac{y_1^2-(m_1+m_2)x_1y_1+m_1m_2x_1^2}{\sqrt{1+(m_1^2+m_2^2)+m_1^2m_2^2}}$$

$$=\frac{y_1^2-\left(-\frac{2h}{b}\right)x_1y_1+\frac{a}{b}x_1^2}{\sqrt{1+\frac{4h^2}{b^2}-\frac{4a}{b}+\frac{a^2}{b^2}}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}$$

Exercise 10

1. Find the angle between the lines :—

(i) $x^2+2xy\cot 2\alpha-y^2=0.$

(ii) $x^2-5xy+4y^2=0.$

(iii) $x^2-7xy+12y^2=0.$

(iv) $x^2+2xy\cot\theta+y^2=0.$

(v) $2x^2+{}^{2}\!\sqrt{17}\,xy+4y^2=0.$ → $2x^2+2\sqrt{17}xy+4y^2=0.$

2. Find the equations of the straight lines bisecting the angles between the pairs of lines :—

(i) $4x^2-16xy+7y^2=0.$

(ii) $33x^2-13xy-6y^2=0.$

(iii) $x^2-2xy\sec\theta+y^2=0.$

(iv) $x^2+2xy\cot\theta+y^2=0.$

3. If pairs of st. lines $x^2+2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each bisects the angles between the other pair, then

$$pq=+1.$$

4. Show that the lines $2x^2+6xy+y^2=0$ are equally inclined to the lines $4x^2+18xy+y^2=0.$

5. Show that the condition that one of the lines $ax^2+2hxy+by^2=0$ may be perpendicular to one of the lines $a'x^2+2h'xy+b'y^2=0$ is

$$(aa'-bb')^2+4(ha'+h'b)(ah'+hb')=0.$$

6. Show that one of the st. lines $ax^2+2hxy+by^2=0$ coincides with one of the straight lines

$$a'x^2+2h'xy+b'y^2=0$$

if $$(ab'-a'b)^2+4(ah'-a'h)(bh'-b'h)=0.$$

7. Show that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is

$$\frac{\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$$

8. Find the equation of the perpendiculars to the lines $ax^2+2hxy+by^2=0$ at the origin.

9. The orthocentre of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is (x',y'), show that

$$\frac{x'}{l}=\frac{y'}{m}=\frac{a+b}{am^2+2ham+bl^2}$$

10. Show that the condition that the two of the straight lines represented by

$$Ax^3+3Bx^2y+3Cxy^2+Dy^3=0$$

may be at right angles is

$$A^2+3AC+3BD+D^2=0.$$

11. Show that the lines given by $ax+by+c=0$ and $(ax+by)^2-3(bx-ay)^2=0$ form the sides of an equilateral triangle.

12. Prove that two of the straight lines given by

$ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$ will be at right angles if

$(b+d)(ad+be)+(a-c)^2(a+c+e)=0$

[**Hint** :—Take $(y^2+hxy-x^2)$ as one of the factors of the equation]

13. Show that the angle between one of the lines given by $ax^2+2hxy+by^2=0$ and one of the lines given by

$$ax^2+2hxy+by^2+k(x^2+y^2)=0$$

is equal to the angle between the other two lines.

14. Prove that the pair of the lines

$$ax^2+2hxy+by^2=0$$

is *equally* inclined to the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0$$

15. If the lines $ax^2+2hxy+by^2=0$ be two sides of a parallelogram, and $lx+my=1$ be one of its diagonals, show that the equation of the other diagonal is

$$y(bl-hm)=x(am-hl).$$

7.7 The necessary and sufficient condition that the general equation of the second degree

$$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$$

may represent a pair of straight lines is

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0}$$

If we succeed in breaking up the given second degree equation into two *linear* factors, we can say that it represents two straight lines.

Let us *assume* that $a \neq 0$. Therefore, on multiplying the given equation by a, and then arranging it in powers of x, we get

$$a^2x^2+2ax(hy+g)=-aby^2-2afy-ac$$

Adding $(hy+g)^2$ to both sides, we get

$$a^2x^2+2ax(hy+g)+(hy+g)^2=y^2(h^2-ab)+2y(gh-af)+(g^2-ac)$$

This gives

$$(ax+hy+g)=\pm\sqrt{y^2(h^2-ab)+2y(gh-af)+(g^2-ac)}$$

From this, we cannot get x in terms of y, involving term of *first* degree, *unless* the quantity under the radical sign is a perfect square.

This requires that

$$(gh-af)^2=(h^2-ab)(g^2-ac)$$

or $$g^2h^2-2afgh+a^2f^2=g^2h^2-abg^2-ach^2+a^2bc$$

or $$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0} \qquad \ldots\ldots(1)$$

This relation when put in the determinant form gives

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0 \quad \ldots(2)$$

Note : *The student is advised to commit results (1) and (2) to memory.*

Alternatively :

The given second degree equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots(1)$$

If this represents two straight lines, then the L. H. S. can be factorized into two linear factors involving x and y.

Thus (1) can be put as

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv(lx+my+n)\,(l'x+m'y+n')=0$$

This gives

$$ll'=a,\ lm'+l'm=2h,\ mm'=b$$

$$ln'+ln=2g,\ mn'+m'n=2f,\ nn'=c$$

Now the required condition can be obtained by eliminating l, m, n, l', m', n' from the above equations. We will do this by adopting the following determinant method :

$$\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}=0$$

or $$\begin{vmatrix} 2ll' & lm'+l'm & ln'+l'n \\ lm'+l'm & 2mm' & mn'+m'n \\ ln'+l'n & lm'+l'm & 2nn' \end{vmatrix}=0$$

Substituting the values for ll', mm', etc. we get

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix}=0$$

or $$8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Now $8\neq0$ $\therefore$ $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

which gives the required condition.

Expanding this determinant, we get

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0}$$

7.8. Given that

$$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$$

represents two straight lines, to find the angle between them.

Let the lines represented by the given equation be

$y=m_1x+c_1$ and $y=m_2x+c_2$

Then

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv b(m_1x-y-c_1)\,(m_2x-y-c_2)$$

Equating coefficients of xy and x^2 on both sides, we get

$$m_1+m_2=-\frac{2h}{b},\quad m_1m_2=\frac{a}{b}$$

Thus angle θ between the lines is given by

$$\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}$$

$$=\pm\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

$$=\pm\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\therefore \qquad \theta=\tan^{-1}\left[\pm\frac{2\sqrt{h^2-ab}}{a+b}\right]$$

Cor. The lines are *perpendicular* if $\mathbf{a+b=0}$

and *parallel* if $\mathbf{ab-h^2=0}$

7.9. Lines joining the origin to the points of a given curve $\phi(\mathbf{x, y})\equiv\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$ and a given line

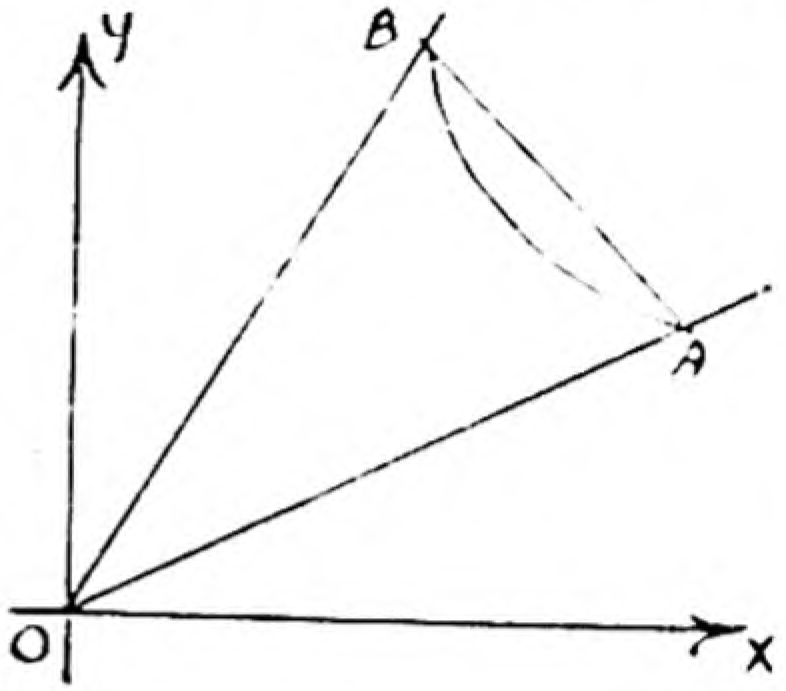
$$\mathbf{lx+my=n}$$

Let the straight line $lx+my=n$ cut the given curve in two points A and B. Now we have to find the joint equation of the straight lines OA and OB, where O is the origin. Such an equation must be homogeneous of second degree in x and y. Therefore, the equation of the given curve, *i.e.*,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots(1)$$

is to be made homogeneous of second degree in x and y with the *help* of

$$lx+my=n \quad \ldots(2)$$



Now (2) can be written as

$$\frac{lx+my}{n}=\mathbf{1} \qquad \text{(Please note this step).}$$

Therefore, equation (1) is made homogeneous of second degree in x, y, with the help of equation (2) as under :—

$$\mathbf{ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{n}\right)+fy\left(\frac{lx+my}{n}\right)+c\left(\frac{lx+my}{n}\right)^2}=0 \quad \ldots(3)$$

(Please note this step)

Then (3) gives the required equation.

Solved Examples

Ex. 1. Prove that the equation

$$12x^2+7xy-10y^2+13x+45y-30=0$$

represents two straight lines, and find the angle between them.

Sol. Here $a=12$, $h=\frac{7}{2}$, $b=-10$,

$g=\frac{13}{2}$, $f=\frac{45}{2}$, $c=-35$

Hence $abc+2fgh-af^2-bg^2-ch^2$

$$=12\times(-10)\times(-35)+2\times\tfrac{45}{2}\times\tfrac{13}{2}\times\tfrac{7}{2}-12(\tfrac{15}{2})^2$$
$$-(-10)(\tfrac{13}{2})^2-(-35)(\tfrac{7}{2})^2$$
$$=-1875+\tfrac{7500}{4}=0$$

Hence the given equation represents two straight lines

Next, if θ be the angle between these lines then,

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\frac{49}{4}+120}}{12-10}$$

$$=\frac{2\sqrt{529/4}}{2}=\frac{23}{2}$$

$$\therefore \qquad \theta=\tan^{-1}(\tfrac{23}{2})$$

Ex. 2. *Prove that the general equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel straight lines if $h^2=ab$ *and* $bg^2=af^2$ *Prove also that the distance between them is*

$$2\sqrt{\frac{g^2-ac}{a(a+b)}} \qquad (K.U.)$$

Sol. Let $\sqrt{a}\,x+\sqrt{b}\,y+c_1=0$ and
$\sqrt{a}\,x+\sqrt{b}\,y+c_2=0$

be the two parallel st. lines represented by the given equation. Therefore, comparing the coefficients, we get

$$\sqrt{a}(c_1+c_2)=2g, \qquad \sqrt{b}(c_1+c_2)=2f$$

$$\therefore \qquad c_1+c_2=\frac{2g}{\sqrt{a}}=\frac{2f}{\sqrt{b}}$$

This gives $bg^2=af^2$

Also, $2\sqrt{ab}=2h$ which gives $ab=h^2$

$c_1c_2=c$

Distance between the parallel lines is

$$\frac{c_1-c_2}{\sqrt{a+b}}=\frac{\sqrt{(c_1+c_2)^2-4c_1c_2}}{\sqrt{a+b}}$$

$$=\frac{\sqrt{\dfrac{4g^2}{a}-4c}}{\sqrt{a+b}}=2\sqrt{\frac{g^2-ca}{a(a+b)}}$$

Ex. 3. Find the equations of the two straight lines represented by

$$x^2+xy-2y^2+x+5y-2=0$$

Sol The given equation can be written as

$$x^2+(y+1)x-(2y^2-5y+2)=0$$

$$\therefore \quad x=\frac{-(y+1)\pm\sqrt{(y+1)^2+4(2y^2-5y+2)}}{2}$$

$$=\frac{-(y+1)\pm3\sqrt{y^2-2y+1}}{2}$$

$$=\frac{-(y+1)\pm3(y-1)}{2}=y-2,\ -2y+1$$

Hence the equations are

$$x=y-2 \quad \text{or} \quad x-y+2=0$$

$$\text{and} \quad x=-2y+1 \quad \text{or} \quad x+2y-1=0$$

Ex. 4. Show that the lines joining the origin to the points common to

$$3x^2+5xy-3y^2+2x+3y=0 \quad \text{and}$$

$$3x-2y=1 \text{ are at right angles.}$$

Sol. Making the first equation homogeneous of second order with the help of the second equation, we have

$$3x^2+5xy-3y^2+2x(3x-2y)+3y(3x-2y)=0$$

or

$$9x^2+10xy-9y^2=0.$$

This gives the equation of the two lines.

The lines will be at right lines, if the sum of the coefficients of x^2 and y^2 is zero. This *is* so, because the coefficient of x^2 is 9 and that of $y^2=-9$.

Exercise 11

Show that the following equations represent two straight lines, and find the angle between them. Find also their points of intersection :—

1. $x^2-5xy+4y^2+4x-y-5=0$.
2. $6y^2-xy-x^2+10y+4=0$.
3. $15y^2-13xy+2x^2+5x-4y-3=0$.
4. $xy-2x-3y+6=0$.
5. Prove that the equation

$$x^2+6xy+9y^2+4x+12y-5=0$$

represents two parallel lines, and the distance between them.

6. Find the value of λ, so that the following equations may represent pairs of straight lines.

(*i*) $x^2+\lambda xy+y^2-5x+7y+6\quad 0$.

(*ii*) $\lambda xy-4x+3y-12=0$.

(*iii*) $2x^2-xy-y^2+\lambda x-6y-9=0$.

7. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, prove that the square of the distance of their point of intersection from the origin is

$$\frac{c(a+b)-f^2-g^2}{ab-h^2}.$$

8. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two st. lines at right angles to each other, show that the square of the distance of their point of intersection from the origin is

$$\frac{f^2+g^2}{b^2+h^2}$$ (*K.U. 1957*)

9. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two st. lines, prove that the product of the perpendiculars from the origin on these lines is

$$\frac{c}{\sqrt{\{(a-b)^2+4h^2\}}}.$$

10. Show that the lines joining the origin to the points of intersection of

$$5x^2+12xy-6y^2+4x-2y+3=0$$

and $x-y=1$ are equally inclined to the axes.

11. If the curve $x^2+y^2+2gx+2fy+c=0$ intercepts on the line $lx+my=1$, a length which subtends a right angle at the origin, show that

$$c(l^2+m^2)+2(gl+fm+1)=0.$$

12. Show that the lines joining the origin to the points common to

$$3x^2+5xy-3y^2+2x+3y=0$$

and $$3x-2y=1$$

are at right angles. (*K.U. 1954*)

13. Prove that the angle between the st. lines joining the origin to the intersection of the straight line $y=3x+2$ with the curve

$$x^2+2xy+3y^2+4x+8y-11=0 \text{ is}$$

$$\tan^{-1}\frac{2\sqrt{2}}{3}.$$

14. Show that the pair of lines joining the origin to the intersection of the st. line $y=mx+c$ and the circle $x^2+y^2=a^2$ are at right angles if $2c^2=a^2(1+m^2)$.

CHAPTER VIII

THE CIRCLE

8.1. Circle. Def. A circle is the locus of a point which moves so that its distance from a fixed point is constant.

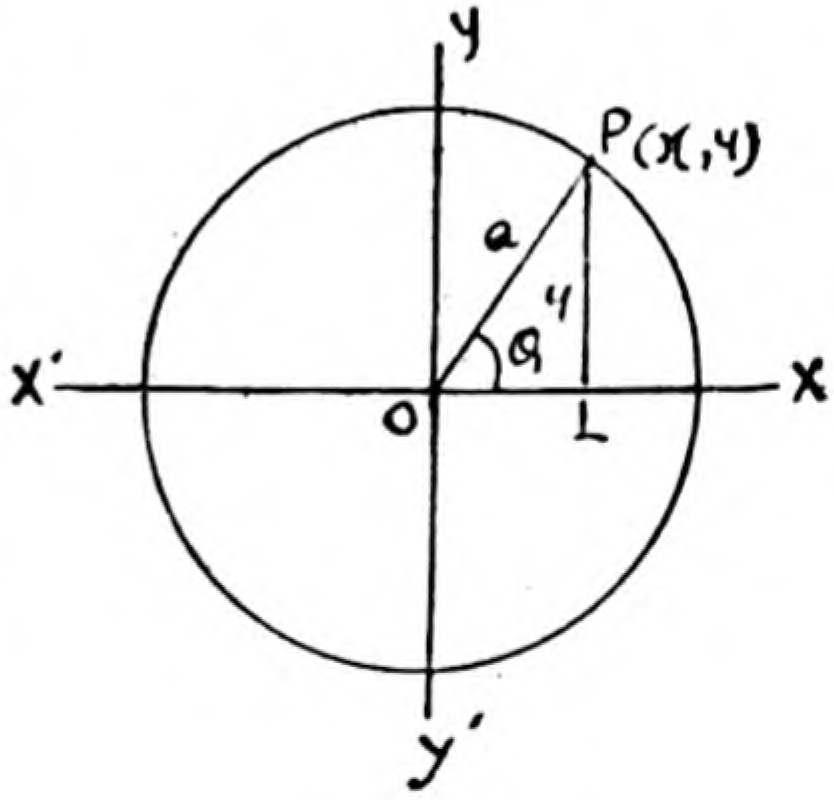
The **fixed** point is called the **centre** of the circle, and the **constant distance** is called the **radius** of the circle.

8.2. Equation of a circle in different forms

(*a*) A circle whose centre is the origin.

To find the equation of a circle in the form $x^2+y^2=a^2$.

Let O be the centre and "a" the radius of the circle. Take O as the origin and XOX′ and YOY′ as the axes.



Let P(x, y) be any point on the circle.

Draw PL ⊥ on OX. Then OL$=x$ and PL$=y$.

Now from the right-angled triangle OLP, we have

$$\text{OL}^2+\text{LP}^2=a^2$$

or

$$\mathbf{x^2+y^2=a^2}$$

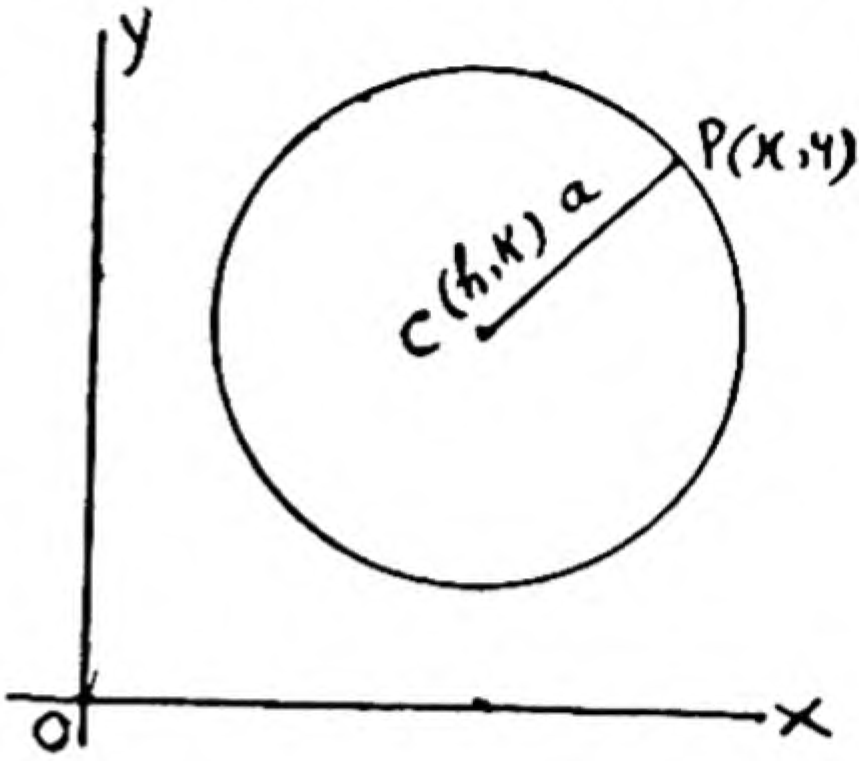
which is the required form.

Note : The above form being the *simplest form* of the equation of a circle, is called the *standard form.*

(*b*) Central form

To find the equation of a circle whose centre and radius are given.

Let C(h, k) be the centre and "a" the radius of the circle.



Take P(x, y) any point on the circle and join CP.

Then CP$=a$.

Now the distance between C and P $= a$

$\therefore \quad \sqrt{(x-h)^2+(y-k)^2}=a$ (distance formula)

Squaring, we get

$$\mathbf{(x-h)^2+(y-k)^2=a^2}$$

which is the required form.

(*c*) **General form**

To prove that $x^2+y^2+2gx+2fy+c=0$ *represents a circle, and to find its centre and radius.*

The equation $x^2+y^2+2gx+2fy+c=0$ can be written as

$$(x^2+2gx)+(y^2+2fy)=-c$$

or $\quad (x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$

(Adding g^2+f^2 to both sides)

or $\quad (x+g)^2+(y+f)^2=g^2+f^2-c$

or $\quad \{x-(-g)\}^2+\{y-(-f)\}^2=(\sqrt{g^2+f^2-c})^2 \quad \ldots(1)$

(Please note this step)

Now (1) is of the form

$$(x-h)^2+(y-k)^2=a^2$$

and, therefore, represents a circle with $(\mathbf{-g, -f})$ as the *centre* and $\sqrt{g^2+f^2-c}$ as *the radius.*

Hence we have the following rule to write down the centre and the radius of a circle whose equation is given in the General Form :—

(*i*) *Write the equation of the circle so that the co-efficients of* x^2 *and* y^2 *are each*$=1$ *on the L.H.S. (the R.H.S. being zero) by dividing throughout, if necessary, by the co-efficient of* x^2 *or* y^2.

(*ii*) *Compare this with* $x^2+y^2+2gx+2fy+c=0$ *and write down the values of*

$g(=\frac{1}{2}$ *co-eff. of* $x)$, $f(=\frac{1}{2}$ *co-eff. of* $y)$, $c(=$*constant term*$)$.

(*iii*) *Then the centre is* $(-\mathbf{g}, -\mathbf{f})$ *and the radius*

$$=\sqrt{\mathbf{g}^2+\mathbf{f}^2-\mathbf{c}}.$$

Note: The circle is *real*, a *point-circle*, or *imaginary* according as

$$\mathbf{g}^2+\mathbf{f}^2-\mathbf{c} \gtreqless 0.$$

Ex. Find the centre and the radius of the circle

$$4x^2+4y^2+12ax-6ay-a^2=0.$$

Sol. Here we have to divide throughout by 4, the coefficient of x^2 or y^2, in order to make the coefficients of x^2+y^2 each equal to 1.

Thus the given equation becomes

$$x^2+z^2+3ax-\frac{3}{2}ay-\frac{a^2}{4}=0 \qquad \text{...(1)}$$

Comparing (1) with $x^2+y^2+2gx+2fy+c=0$, we get

$$g=\frac{1}{2}(3a)=\frac{3a}{2},\ f=\frac{1}{2}\left(-\frac{3a}{2}\right)=-\frac{3a}{4}$$

$$c=-\frac{a^2}{4}$$

∴ the co-ordinates of the centre are :

$$\left(-\frac{3a}{2}, \frac{3a}{4}\right) \qquad (\because\ -g, -f)$$

and the radius $=\sqrt{\frac{9a^2}{4}+\frac{9a^2}{16}+\frac{a^2}{4}} \qquad (\because\ \sqrt{g^2+f^2-c})$

$$=\frac{7a}{4}.$$

Alternative Method :

The given equation of the circle can be put as

$$x^2+y^2+3ax-\frac{3a}{2}\ y-\frac{a^2}{4}=0$$

or $$(x^2+3ax)+\left(y^2-\frac{3a}{2}y\ \right)=\frac{a^2}{4}$$

or $$\left\{\left(\ x^2+3ax+\frac{9a^2}{4}\right)\right\}+\left\{\left(y^2-\frac{3a}{2}\ y+\frac{9a^2}{16}\ \right)\right\}$$
$$=\frac{a^2}{4}+\frac{9a^2}{4}+\frac{9a^2}{16}$$

(Please note this step)

or $$\left(\ x+\frac{3a}{2}\right)^2+\left(\ y-\frac{3a}{4}\ \right)^2=\left(\frac{7a}{4}\ \right)^2.$$

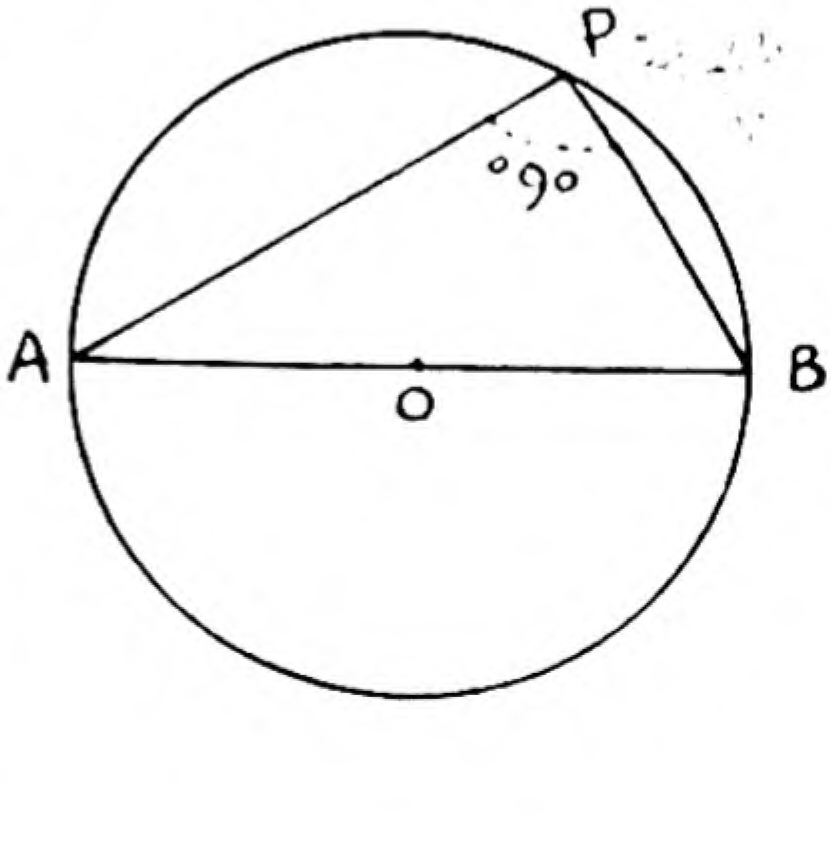
This is of the type $(x-h)^2+(y-k)^2=a^2$ and, as such, represents a circle whose centre is $\left(\frac{3a}{2},\ \frac{-3a}{4}\right)$ and radius

$$=\frac{7a}{4}.$$

(*d*) Diameter Form

To find the equation of the circle on the join of (x_1, y_1) *and* (x_2, y_2) *as diameter.*

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the ends of the diameter AB. Take $P(x, y)$ any other point on the circle. Join PA and PB.



Now, $\angle APB=90°$

(∵ angle in a semi-circle)

∴ AP ⊥ on PB ...(1)

The slope of AP$=\dfrac{y-y_1}{x-x_1}$

$$\left(\because\ \frac{y_2-y_1}{x_2-x_1}\right)$$

The slope of PB $=\frac{y-y_2}{x-x_2}$ (1) gives

$$\frac{y-y_1}{x-x_1}, \frac{y-y_2}{x-x_2}=-1 \qquad (\because\ m_1m_2=-1)$$

or $$(y-y_1)(y-y_2)=-(x-x_1)(x-x_2)$$

or $$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$$

which gives the *required* form.

(*e*) Three-point form

To find the equation of a circle passing through three points.

Let $x^2+y^2+2gx+2fy+c=0$ be the equation of the circle passing through three *given* points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Now (x_1, y_1) lies on the circle

$\therefore$ $$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad ...(1)$$

Similarly, $$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad ...(2)$$

and $$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \qquad ...(3)$$

By solving equations (1), (2), and (3) for g, f, c we can get the equation of the circle after these values of g, f, c are substituted in $x^2+y^2+2gx+2fy+c=0$.

8.3. Characteristics of the equation of the circle

$$\mathbf{x^2+y^2+2gx+2fy+c=0.}$$

(1) *It is an equation of the second degree in x and y.*

(2) *The coefficients of x^2 and y^2 are equal.*

(3) *There is no term containing the product xy.*

Solved Examples

Ex. 1. Find the equation of the circle whose centre is $(-5, -6)$ and radius is 10.

Sol. We know that the equation of the circle whose centre is (h, k) and radius is "a" is given by

$$(x-h)^2+(y-k)^2=a^2.$$

Here $$h=-5,\ k=-6, \text{ and } a=10$$

$\therefore$ the required equation is

$$(x+5)^2+(y+6)^2=10^2$$

or $$x^2+y^2+10x+12y-39=0.$$

Ex. 2. Find the equation of the circle which passes through the centre of the circle $x^2+y^2+8x+10y-7=0$ and is concentric with the circle $x^2+y^2-4x-6y=0$.

Sol. The centre of the circle $x^2+y^2+8x+10y-7=0$ is $(-4, -5)$, and that of $x^2+y^2-4x-6y=0$ is $(2, 3)$.

Now the required circle passes through $(-4, -5)$ and has the same centre (concentric) as

$$x^2+y^2-4x-6y=0 \text{ i.e., } (2, 3).$$

Let "a" be its radius

$\therefore$ the equation becomes

$$(x-2)^2+(y-3)^2=a^2 \qquad \ldots(1)$$

Now (1) passes through $(-4, -5)$

$\therefore$ $$(-4-2)^2+(-5-3)^2=a^2$$

or $$36+64=a^2$$

which gives $$a=10.$$

Hence (1) becomes

$$(x-2)^2+(y-3)^2=100.$$

Ex. 3. Find the equation of the circle the extremities of whose diameter are $(2, 3)$ and $(5, 4)$.

Sol. Here $x_1=2, y_1=3\,;\, x_2=5, y_2=4$

$\therefore$ $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \qquad \text{(Article 8.2)}$$

gives $$(x-2)(x-5)+(y-3)(y-4)=0$$

or $$x^2+y^2-7x-7y+22=0$$

which gives the required equation.

Ex. 4. Find the equation of the circle which passes through the points $(-3, 4)$, $(2, 0)$, and $(1, 5)$. Find the co-ordinates of the centre and the radius of this circle.

Sol. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad ...(i)$$

Since it passes through $(-3, 4)$, we have

$$9+16-6g+8f+c=0$$

or

$$-6g+8f+c=-25 \qquad ...(ii)$$

Similarly, by substituting the points (2, 0) and (1, 5) in (*i*) we get

$$4-4g+c=0 \qquad ...(iii)$$

and

$$2g+10f+c=-26 \qquad ...(iv)$$

Solving equations (*ii*), (*iii*), and (*iv*) simultaneously, we get

$$g=\tfrac{1}{2}, f=-\tfrac{5}{2}, c=-2.$$

Substituting these values of g, f, and c in (*i*) we get

$$x^2+y^2+2(\tfrac{1}{2})x+2(-\tfrac{5}{2})y-2=0$$

or

$$x^2+y^2+x-5y-2=0$$

which gives the required equation

(*ii*) Next $g=\tfrac{1}{2}, f=-\tfrac{5}{2}$

∴ the co-ordinates of the centre are

$$(-\tfrac{1}{2}, \tfrac{5}{2}) \qquad (\because \ -g, -f)$$

and

$$\text{radius}=\sqrt{g^2+f^2-c}=\sqrt{\tfrac{1}{4}+\tfrac{25}{4}+2}$$

$$=\frac{\sqrt{34}}{2}$$

Ex. 5. Find the equation of the circle passing through the points (4, 1) and (6, 5) whose centre lies on the line

$$4x+y-16=0$$

Sol. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad ...(i)$$

Since (1) passes through (4, 1) and (6, 5)

∴

$$8g+2f+c=-17 \qquad ...(ii)$$

and

$$12g+10f+c=-61 \qquad ...(iii)$$

Again, the centre $(-g, -f)$ of the circle lies on the line

$$4x+y-16=0$$

∴

$$4g+f+16=0 \qquad ...(iv($$

Solving (*ii*), (*iii*) and (*iv*) simultaneously, we get

$$g=-3, f=-4, c=15.$$

Substituting these values of g, f and c in (*i*) we get

$$x^2+y^2-6x-8y+15=0$$

which gives the required equation.

Note : *The equation of the circle*

$$x^2+y^2+2gx+2fy+c=0$$

contains three constants g, f *and* c.

Three conditions are, therefore, required to find the equation of a circle. Thus the equation of the circle can be found if :—

(*i*) the circle passes through any three given points,

(*ii*) the circle passes through any two given points and its centre lies on

(*a*) the x-axis (*b*) the y-axis

(*c*) any given line.

and so on.

Exercise 12

Find the radius and the co-ordinates of the centre of the following circles :—

1. $x^2+y^2-8x-16y+78=0.$
2. $7x^2+7y^2-4x-y=3.$
3. $ax^2+ay^2=bx+cy.$
4. $x^2+y^2-4x-8y-41=0.$
5. $3x^2+3y^2-5x-6y+4=0.$
6. $x^2+y^2=k(x+k).$
7. $x^2+y^2=2gx-2fy.$

Find the equation to the circle

8. whose radius is 3 and whose centre is $(-1, 2)$.
9. whose radius is 10 and whose centre is $(-5, -6)$.
10. whose radius is $\sqrt{a^2-b^2}$ and whose centre is $(-a, -b)$.
11. whose radius is $a+b$ and whose centre is $(a, -b)$.

Find the equation to the circle passing through the points

12. $(0, 0), (a, 0), (0, b)$

13. $(1, 2). (3, -4), (5, -6)$ (J & K U) (*J* & *K*)

14. $(1, 1), (2, -1). (3, 2)$

15. $(a, b), (a, -b), (a+b, a-b)$

16. Prove that the equation of the circle circumscribing the triangle formed by the lines

$$x+y=6, \qquad 2x+y=4, \qquad \text{and} \qquad x+2y=5$$

$$\therefore \quad x^2+y^2-17x-19y+50=0$$

17. Find the equation of the circle which passes through the points $(4, 2)$, $(-6, -2)$ and has its centre on the y-axis.

18. Find the equation of the circle which passes through the points $(1, -2)$ and $(4, -3)$ and whose centre lies on the line

$$3x+4y=7$$

19. Find the equation of the circle which passes through $(4, 1)$ and $(6, 5)$ and whose centre lies on the line $4x+y-16=0$

20. Find the equation of the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the axes.

21. Find the equation of the circle which passes through the origin and cuts off intercepts equal to h and K from the positive parts of the axes.

Find the equation to the circle which

22. touches each axis at a distance 5 from the origin.

23. touches each axis and is of radius "a".

Find the equation of the circle the co-ordinates of the extremities of whose diameter are

24. $(0, 1)$ and $(1, 1)$

25. (a, b) and (c, d)

26. $(-2, -3)$ and $(5, 4)$

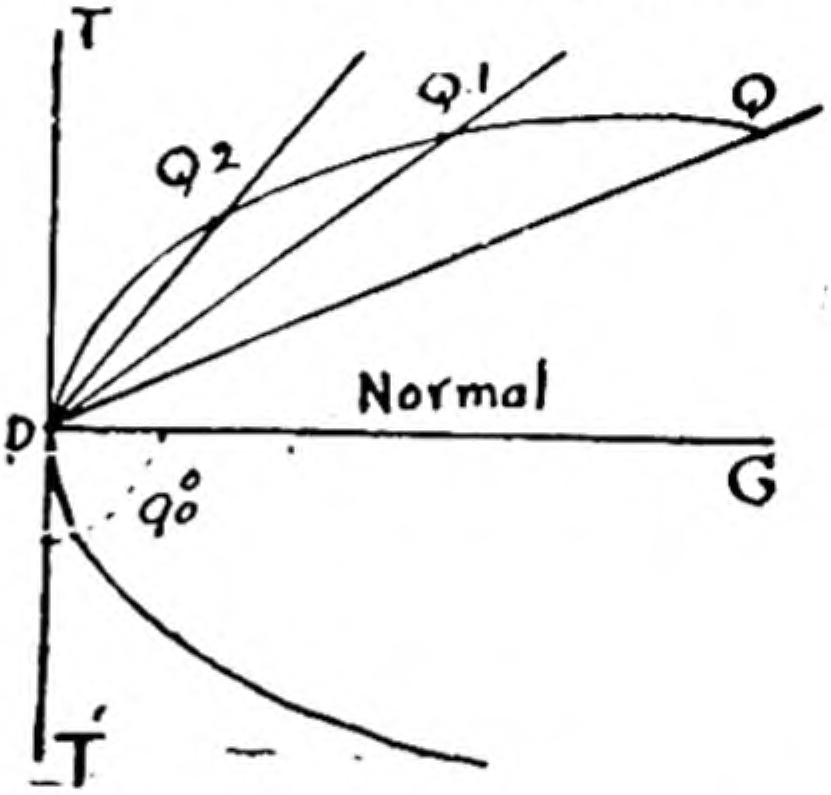
27. $(1, 1)$ and $(5, -1)$

28. $(3, 4)$ and $(2, -7)$

CHAPTER IX

TANGENTS AND NORMALS

9.1. Tangent. Def. Let the two neighbouring points P and Q be taken on a curve. Join PQ; then PQ is called a *secant*. Let the point Q start moving along the curve towards P taking positions $Q_1, Q_2, \ldots$ Then TPT_1 which is the *limiting* position of the line PQ when Q moving along the curve *ultimately coincides* with P is called the **tangent** to the curve at P.



9.2. Normal. Def. *The line PG through P and perpendicular to the tangent TPT_1 is called the* **normal** *to the curve at the point P.*

9.3. Equation of the tangent.

To find the equation of the tangent at the point (x_1, y_1) of the circle

$$x^2+y^2=a^2.$$

Let $P(x_1, y_1)$ be the given point and $Q(x_2, y_2)$ any other point lying on the given circle.

$$\therefore \qquad x_1^2+y_1^2=a^2 \qquad \ldots(i)$$

$$\text{and} \qquad x_2^2+y_2^2=a^2 \qquad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)=0$$

$$\text{or} \qquad (x_2+x_1)(x_2-x_1)+(y_2+y_1)(y_2-y_1)=0$$

$$\text{or} \qquad (y_2+y_1)(y_2-y_1)=-(x_2+x_1)\,x_2-x_1)$$

or
$$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1} \qquad \ldots(iii)$$

Now the equation of PQ is
$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \ldots(iv)$$

Substituting $-\frac{x_2+x_1}{y_2+y_1}$ for $\frac{y_2-y_1}{x_2-x_1}$ [as obtained in (*iii*)] in (*iv*), we get
$$y-y_1=-\frac{x_2+x_1}{y_2+y_1}(x-x_1) \qquad \ldots(v)$$

Equation (*v*) will give the equation of the secant PQ. This secant will become the tangent at P, if Q moves up and *ultimately coincides* with P, *i.e.*, if $x_2 \to x_1$ and $y_2 \to y_1$. Thus in such a case the equation (*v*) becomes
$$y-y_1=-\frac{2x_1}{2y_1}(x-x_1)$$

or
$$yy_1-y_1^2=-xx_1+x_1^2$$

or
$$xx_1+yy_1=x_1^2+y_1^2=a^2 \qquad [\because \text{ of } (i)]$$

Hence
$$\mathbf{xx_1+yy_1=a^2}$$

gives the required equation of the tangent at the point $P(x_1, y_1)$.

9.4. Equation of the Normal.

To find the equation of the normal at the point (x_1, y_1) *of the circle* $x^2+y^2=a^2$.

The equation of tangent at (x_1, y_1) is
$$xx_1+yy_1=a^2. \qquad \ldots(i)$$

$$\text{Slope of } (i) = -\frac{x_1}{y_1}$$

$$\therefore \quad \text{slope of the normal} = \frac{y_1}{x_1}$$

($\because$ the normal is $\perp$ to (*i*) and $m_1m_2=-1$)

$\therefore$ the equation of the normal is

$$y-y_1=\frac{y_1}{x_1}(x-x_1) \;\ldots\ldots \qquad [\because\; y-y_1=m(x-x_1)]$$

or $\qquad x_1y-xy_1=0$

or $$\frac{\mathbf{x}}{\mathbf{x}_1}=\frac{\mathbf{y}}{\mathbf{y}_1}$$

which is the required equation.

Note : (1) The equation of the normal shows that it passes through (0, 0) which is the centre of the circle

$$x^2+y^2=a^2$$

Note : (2) The point (x_1, y_1) is called the *foot* of the normal.

9.5. *To find the equation of the tangent at the point* (x_1, y_1) *of the circle*

$$x^2+y^2+2gx+2fy=c.$$

Let $P(x_1, y_1)$ be the given point and $Q(x_2, y_2)$ any other point on the curve. Then

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \ldots(i)$$

and $$x_2^2+y_3^2+2gx_2+2fy_2+c=0 \qquad \ldots(ii)$$

Subtracting (*i*) from (*ii*), we get

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0$$

or $$(x_2+x_1)(x_2-x_1)+(y_2+y_1)(y_2-y_1)+2g(x_2-x_1)+2f(y_2-y_1)=0$$

or $$[(x_2+x_1)(x_2-x_1)+2g(x_2-x_1]+[(y_2+y_1)(y_2-y_1)+2f(y_2-y_1)]=0$$

(Please note this step)

or $$(x_2-x_1)(x_1+x_2+2g)+(y_2-y_1)(y_1+y_2+2f)=0$$

or $$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_1+x_2+2g}{y_1+y_2+2f} \qquad \ldots(iii)$$

Now the equation of PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x_2-x_1) \qquad \ldots(iv)$$

Equation (*iv*) will become the equation of secant PQ if we substitute $-\dfrac{x_1+x_2+2g}{y_1+y_2+2f}$ for $\dfrac{y_2-y_1}{x_2-x_1}$ [obtained in (*iii*)] in it.

Thus the equation of the secant PQ is

$$y-y_1=-\frac{x_1+x_2+2g}{y_1+y_2+2f}\ (x-x_1) \qquad ...(v)$$

Again the secant PQ will become the tangent at P if Q moves up and *ultimately coincides* with P, *i.e.*, $x_2 \to x_1$ and $y_2 \to y_1$. In such a case equation (*v*) becomes

$$y-y_1=-\frac{2(x_1+g)}{2(y_1+f)}\ (x-x_1)$$

or $\quad yy_1-y_1^2+fy-fy_1=-xx_1+x_1^2-gx+gx_1$

or $\quad xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$

(Please note this step)

Adding (gx_1+fy_1+c) to both sides, we get

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c=0$$

[Making use of (*i*)]

Hence $\quad \mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}$
gives the required equation.

9.6. *To find the equation of the normal at the point* (x_1, y_1) *of the circle*

$$x^2+y^2+2gx+2fy+c=0$$

The equation of the tangent at (x_1, y_1) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

(Article 9.5)

This can be written as

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0 \qquad ...(1)$$

Slope of (*i*) $\quad =-\dfrac{x_1+g}{y_1+f}$

But the normal is $\perp$ to the tangent

$\therefore$ the slope of the normal $= \dfrac{y_1+f}{x_1+g}$

$\therefore$ the equation of the normal at (x_1, y_1) is

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$

or $\quad \mathbf{y(x_1+g)-x(y_1+f)+fx_1-gy_1=0}$

which is the required equation.

Note : Important

(1) **Rule to write down the equation of the tangent at the point $(\mathbf{x_1}, \mathbf{y_1})$ of a circle.**

In the equation of the circle, change x^2 *to* xx_1, y^2 *to* yy_1, x *to* $\frac{1}{2}(x+x_1)$, *and* y *to* $\frac{1}{2}(y+y_1)$.

If the numerical values of x_1 and y_1 are given, substitute them in the above result.

(2) **Rule to write down the equation of the normal at the point $(\mathbf{x_1}, \mathbf{y_1})$ of a circle :**

(*i*) *Write down the equation of the tangent at* (x_1, y_1) *and find the slope of this tangent.*

(*ii*) *Find the slope of the normal.*

This will be $= -\dfrac{1}{\textit{slope of the tangent}}$

$$\left(m_2=-\frac{1}{m_1}\right)$$

(*iii*) *Write down the equation of the normal at* (x_1, y_1) *by using*

$$y-y_1=m(x-x_1)$$

A Circle and a Line

9.7. **Points of intersection.** *To find the points of intersection of the line* $y=mx+c$ *with the circle* $x^2+y^2=a^2$

The equation of the circle is

$$x^2+y^2=a^2 \qquad \ldots(1)$$

The equation of the line is

$$y=mx+c \qquad \ldots(2)$$

Now in order to find the points of intersection of the line with the circle, we solve (1) and (2) *simultaneously*. The process is given as follows :—

Substituting the value of y from (2) in (1) we get

$$x^2+(mx+c)^2=a^2$$

or $$(1+m^2)x^2+2mc.x+(c^2-a^2)=0 \qquad \ldots(3)$$

Equation (3) being a *quadratic* in x gives two values of x. Let these be x_1 and x_2.

Substituting x_1 and x_2 for x in (2), we get y_1 and y_2, the corresponding values of y, as under :—

$$y_1=mx_1+c \qquad \text{and} \qquad y_2=mx_2+c.$$

Thus (x_1, mx_1+c) and (x_2, mx_2+c) are the points of intersection of the line $y=mx+c$ with the circle $x^2+y^2=a^2$.

Note : *The point (or points) of intersection of a straight line with a circle is always obtained by solving simultaneously the equation of the st. line and that of the circle. Similar method is adopted in the case of other curves as well.*

9.8. Length of the intercept

To find the length of the intercept made by the circle $x^2+y^2=a^2$ *on the line* $y=mx+c$.

Let the line $y=mx+c$ meet the circle in A and B.

From O, draw OL ⊥ on AB. Then AL=LB.

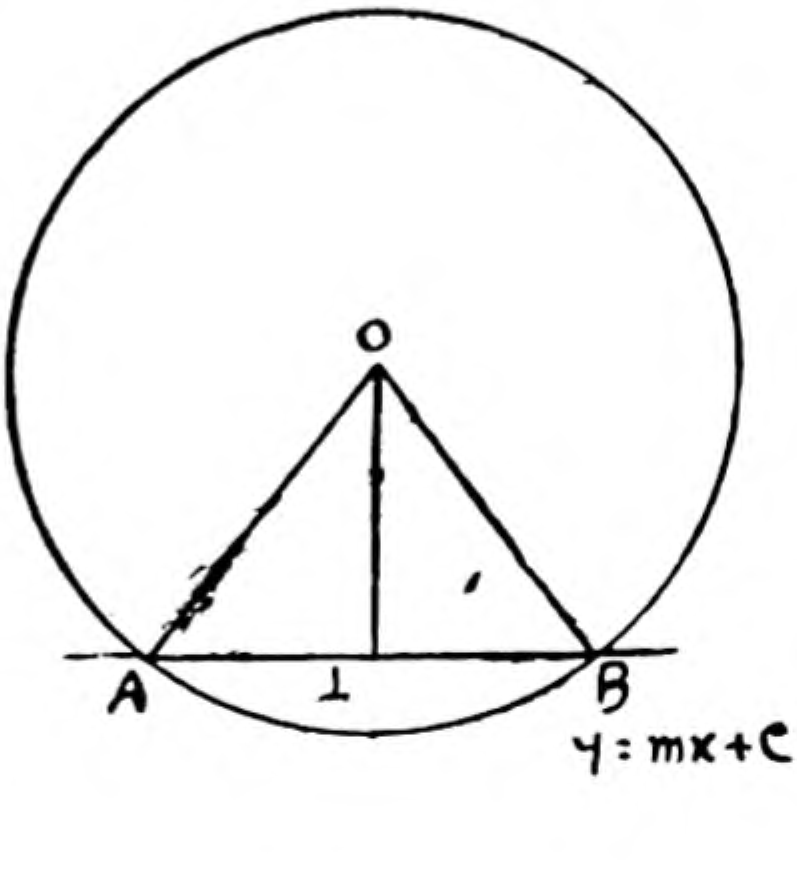
Join OA and OB.

Now from the right-angled triangle ALO,

$$AL=\sqrt{OA^2-OL^2}$$

But OA$=a$, Bashid

OL=length of the perpendicular drawn on the line $y=mx+c$ from (0, 0)



$$=\frac{c}{\sqrt{1+m^2}}$$ (Article 6.1)

$$\therefore \quad AL=\sqrt{a^2-\frac{c^2}{1+m^2}}.$$

Hence AB=the length of the intercept=2AL

$$=2\sqrt{\mathbf{a}^2-\frac{\mathbf{c}^2}{\mathbf{1+m}^2}}.$$

9.9. Condition of tangency.

To find the condition that the line $y=mx+c$ *may touch the circle* $x^2+y^2=a^2$.

First Method : (*Method of Intersection*)

The equation of the line is $y=mx+c$ ...(1)

and that of the circle is $x^2+y^2=a^2$ ...(2)

In order to find the points of intersection of the line with the circle, we will substitute the value of y from (1) in (2).

This will give

$$x^2+(mx+c)^2=a^2$$

or $$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \qquad ...(3)$$

This is a quadratic in x, and the line will touch the circle if the two roots of (3) are *coincident*, which requires its discriminant to vanish.

$$\therefore \quad 4m^2c^2-4(1+m^2)(c^2-a^2)=0$$

or $$m^2c^2-(c^2-a^2+m^2c^2-m^2a^2)=0$$

or $$c^2=a^2(1+m^2)$$

$$\therefore \quad \mathbf{c=\pm a\sqrt{1+m^2}}$$

which is the required condition.

Second Method : (*Intercept Method*)

In Article 9.8, we have seen that the length of the intercept made by the circle $x^2+y^2=a^2$ on the line $y=mx+c$ is

$$2\sqrt{a^2-\frac{c^2}{1+m^2}}.$$

Now if the line $y=mx+c$ touches the circle $x^2+y^2=a^2$, the length of this intercept must be equal to *zero*.

$$\therefore \quad 2\sqrt{a^2-\frac{c^2}{1+m^2}}=0$$

or $$c^2=a^2(1+m^2) \qquad (\because \; 2\neq 0)$$

$$\therefore \quad \mathbf{c=\pm a\sqrt{1+m^2}}$$

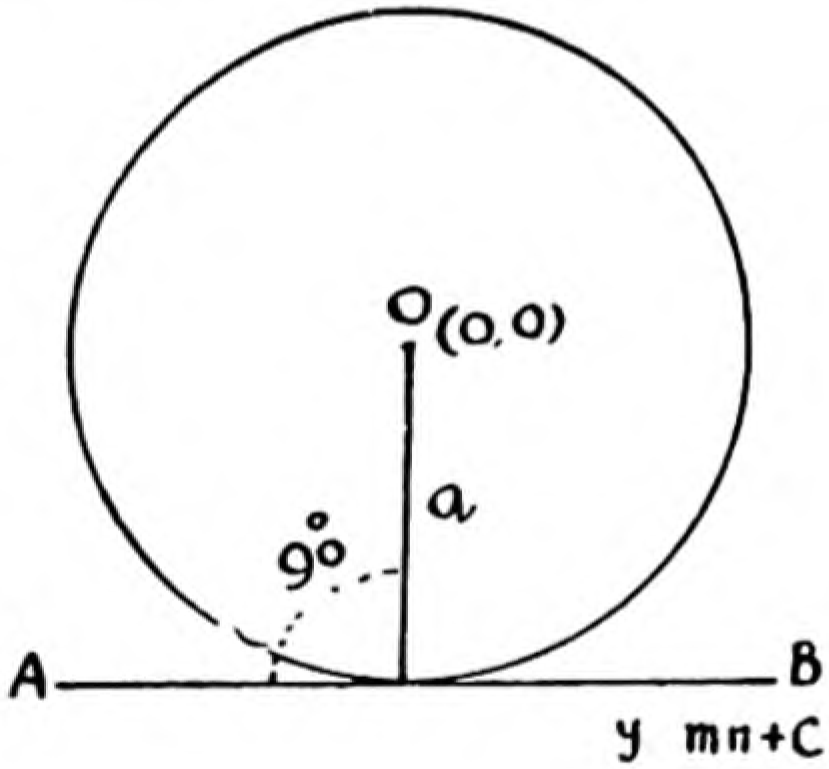
which is the required condition.

Third Method : (*Method of tangent-property*)

If the line touches the circle, then the length of the perpendicular drawn from the centre (0, 0) of the circle on it must be equal to the radius of the circle.

$$\therefore \quad \pm\frac{c}{\sqrt{1+m^2}}=a$$

(Article 6.1)



or $$\mathbf{c=\pm a\sqrt{1+m^2}}$$

which is the required condition.

Note : *In order to find whether a particular line touches a given circle, the third method should be preferred. This method will not, however, give us the point of contact. In such a case the student should adopt the first method.*

9.10. *We have seen above that in order that the line* $\mathbf{y=mx+c}$ *may touch the circle* $\mathbf{x^2+y^2=a^2}$, *c must be equal to* $\pm\mathbf{a}\sqrt{\mathbf{1+m^2}}$.

Hence the equations of the tangents to the circle $\mathbf{x^2+y^2=a^2}$ *in the* **slope** *form are*

$$\mathbf{p=mx\pm a\sqrt{1+m^2}}$$

Note : *The student must bear in mind that the methods for finding equations of tangents explained in Articles 9.3 and 9.5 have to be adopted if the point of contact is given. Otherwise, he can very safely suppose the lines* $\mathbf{y=mx\pm a\sqrt{1+m^2}}$ *to be the tangents to the circle and evaluate "m" with the help of the conditions given in the question.* This is illustrated below :—

Illustration : (1) Find the equation of the tangent to the circle $x^2+y^2=9$ at the point (3,2)

Sol. Here the point of contact is given.

i.e. $x_1=3,\ y=2$

$\therefore$ changing x^2 to $3x$ and y^2 to $2y$ the required equation becomes

$$3x+2y=9 \qquad (\because xx_1+yy_1=a^2)$$

Illustration : (2) Find the equations of the tangents to the circle $\mathbf{x^2+y^2=25}$ which are inclined at an angle 30° to the axis of x,

Sol. The equation of the circle is

$$x^2+y^2=25$$

Here $a^2=25 \quad \therefore a=5$

Now that we have not been given the point of contact, we can find the equations of the tangents as under :

Let $y=mx\pm a\sqrt{1+m^2}$...(1) be two equations of the tangents

$$a=5\ ;\ \ m=\tan 30°=\frac{1}{\sqrt{3}}$$

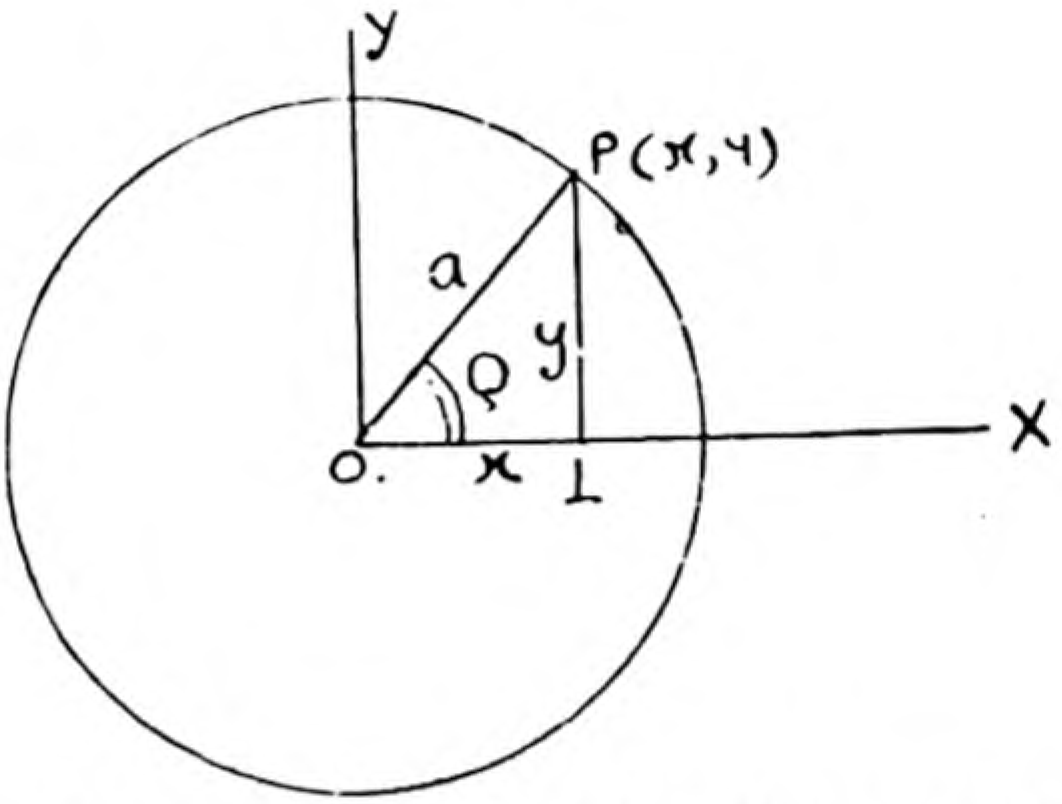
$\therefore$ (1) becomes

$$y=\frac{1}{\sqrt{3}}x\pm5\sqrt{1+\frac{1}{3}}$$

$$=\frac{1}{\sqrt{3}}x\pm5\left(\frac{2}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}x\pm\frac{10}{\sqrt{3}}$$

or $x-\sqrt{3}+10=0$ and $x-\sqrt{3}\,y-10=0$ are the required equations of the two tangents.

9.11. Parametric Form of the equation of the circle $\mathbf{x^2+y^2=a^2}$



Here (0, 0) is the centre of the circle and "a" the radius. Let P (x, y) be *any* point on the circle. Join OP and draw PL ⊥ on OX. Let ∠XOP=θ. Then OP$=a$

Now from the right-angled triangle OLP

$$\text{OL}=\text{X}=\text{OP}\cos\theta=a\cos\theta$$

and $$\text{PL}=y=\text{OP}\sin\theta=a\sin\theta.$$

Hence the co-ordinates of any point P on ths circle $\mathbf{x^2+y^2=a^2}$ *are* ($\mathbf{a}\cos\theta$, $\mathbf{a}\sin\theta$) *and the two equtions* $\mathbf{x}=\mathbf{a}\cos\theta$, $\mathbf{y}=\mathbf{a}\sin\theta$ *taken together are known as the* **Parametric Equations** *of the circle* $\mathbf{x^2+y^2=a^2}$

Note : *The point P* ($a\cos\theta$, $a\sin\theta$) *is briefly written as* "θ". *Thus the point* "θ_1", *will mean* ($a\cos\theta_1$, $a\sin\theta_1$) *and* "θ_2" *will mean* ($a\cos\theta_2$, $a\sin\theta_2$).

9.12. *To find the equation of the chord joining the two points* "θ_1" *and* "θ_2" *on the circle* $x^2+y^2=a^2$.

The two points "θ_1" and "θ_2" are ($a\cos\theta_1$, $a\sin\theta_1$) and ($a\cos\theta_2$, a $\sin\theta_2$)

Hence the equation of the chord joining these two points is

$$y-a\sin\theta_1=\frac{a\sin\theta_2-a\sin\theta_1}{a\cos\theta_2-a\cos\theta_1}\left(x-a\cos\theta_1\right)$$

$$=\frac{\sin\theta_2-\sin\theta_1}{\cos\theta_2-\cos\theta_1}\left(x-a\cos\theta_1\right)$$

$$=\frac{2\cos\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}}{-2\sin\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}}(x-a\cos\theta_1)$$

or $$y-a\sin\theta_1=\frac{\cos\frac{\theta_1+\theta_2}{2}}{-\sin\frac{\theta_1+\theta_2}{2}}(x-a\cos\theta_1)$$

or $$-y\sin\frac{\theta_1+\theta_2}{2}+a\sin\theta_1.\sin\frac{\theta_1+\theta_2}{2}$$

$$=x\cos\frac{\theta_1+\theta_2}{2}-a\cos\theta_1.\cos\frac{\theta_1+\theta_2}{2}$$

or $$x\cos\frac{\theta_1+\theta_2}{2}+y\sin\frac{\theta_1+\theta_2}{2}=a\cos\theta_1.\cos\frac{\theta_1+\theta_2}{2}$$

$$+a\sin\theta_1.\sin\frac{\theta_1+\theta_2}{2}$$

$$=a\cos\left(\theta_1-\frac{\theta_1+\theta_2}{2}\right)$$

$$=a\cos\frac{\theta_1-\theta_2}{2}$$

Hence the required equation of the chord is

$$x\cos\frac{\theta_1+\theta_2}{2}+y\sin\frac{\theta_1+\theta_2}{2}=a\cos\frac{\theta_1-\theta_2}{2}$$

Cor. *The above equation becomes the equation of the tangent at* "θ_1" *if* $\theta_2 \to \theta_1$.

Thus $$\mathbf{x \cos \theta_1 + y \sin \theta_1 = a}$$

is the equation of the tangent to the circle $\mathbf{x^2+y^2=a^2}$ *at the point* "θ_1".

Note: *The equation of the normal to the circle* $x^2+y^2=a^2$ *at the point* "θ" *is*

$$\mathbf{x \sin \theta - y \cos \theta = a.}$$

Solved Examples

Ex. 1. Obtain the equations of the tangent and normal to the circle

$$x^2+y^2-2x-10y+21=0$$

at the point $(-1, 4)$.

Sol. Here $x_1=-1$ and $y_1=4$

$\therefore$ changing x^2 to $xx_1=-x$, y^2 to $yy_1=4y$, x to $\frac{1}{2}(x+x_1)=\frac{1}{2}(x-1)$, and y to $\frac{1}{2}(y+y_1)=\frac{1}{2}(y+4)$,

we get the equation of the tangent at $(-1, 4)$ as

$$-x+4y-2.\tfrac{1}{2}(x-1)-10.\tfrac{1}{2}(y+4)+21=0$$

or $$-x+4y-(x-1)-5(y+4)+21=0$$

or $$-2x-y+2=0$$

or $$2x+y=2$$

which is the equation of the tangent.

Now slope of the tangent$=-2$ $\left(-\frac{a}{b}\right)$

$\therefore$ slope of the normal$=\frac{1}{2}$ $\left(\because\ m_2=-\frac{1}{m_1}\right)$

$\therefore$ Equation of the normal at $(-1, 4)$ is

$$y-4=\tfrac{1}{2}(x+1)$$

or $$2y-8=x+1$$

or $$x-2y+9=0.$$

Ex. 2. Find the condition that the line $lx+my+n=0$ may touch the circle $x^2+y^2=a^2$. Find the point of contact, if the condition is satisfied. *(K. U. 1956)*

Sol. The equation of the line is

$$lx+my+n=0 \qquad \ldots(1)$$

and that of the circle is

$$x^2+y^2=a^2 \qquad \ldots(2)$$

Let (x_1, y_1) be the point of contact, the equation of the tangent at (x_1, y_1) to the circle is

$$xx_1+yy_1=a^2 \qquad \ldots(3)$$

Now (3) is the same as (1)

$\therefore$ comparing coefficients, we get

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{-a^2}{n}$$

$$\therefore \qquad x_1=-\frac{la^2}{n} \text{ and } y_1=-\frac{ma^2}{n}$$

Now (x_1, y_1) lies on (2)

$$\therefore \qquad x_1{}^2+y_1{}^2=a^2 \qquad \ldots(4)$$

Substituting $-\dfrac{la^2}{n}$ and $\dfrac{-ma^2}{n}$ for x_1 and y_1 in (4) respectively, we get

$$\frac{l^2a^4}{n^2}+\frac{m^2a^4}{n^2}=a^2$$

or $$(l^2+m^2)a^2=n^2$$

which gives the required condition and the point of contact is

$$\left(-\frac{la^2}{n},\ -\frac{ma^2}{n}\right).$$

Ex. 3. Show that the line $3x+4y+20=0$ touches the circle $x^2+y^2=16$, and find the point of contact also. *(P.U.)*

Sol. The equation of the line is

$$3x+4y+20=0 \qquad \ldots(1)$$

and that of the circle is

$$x^2+y^2=16 \qquad \ldots(2)$$

Substituting the value of y $\left(=-\dfrac{3x+20}{4}\right)$ from (1) in (2), we get

$$x^2+\frac{(3x+20)^2}{16}=16$$

or

$$25x^2+120x+144=0. \qquad \ldots(3)$$

If (1) touches (2), the roots of (3) must be coincident, *i.e.*, the discriminant of (3) must be zero.

Now the discriminant is $(120)^2-4.25.144$ which is evidently equal to zero.

Hence (1) touches (2).

Also each of the equal roots of (3) $=-\frac{12}{5}$

$\therefore$ the line touches the circle at the point where

$$x=-\tfrac{12}{5}.$$

Substituting this value of x in

$$y=-\frac{3x+20}{4}$$

we get

$$y=-\tfrac{16}{5}.$$

Hence the point of contact is $(-\frac{12}{5}, -\frac{16}{5})$.

Ex. 4. Find the equations of the tangents to the circle

$$x^2+y^2-2x-4y-4=0$$

which are perpendicular to the line

$$3x-4y-1=0$$ *(P.U. 1955)*

Sol. The equation of any line $\perp$ to the line

$$3x-4y-1=0 \text{ is}$$

$$4x+3y+k=0 \qquad \ldots(1)$$

Now if (1) touches the circle

$$x^2+y^2-2x-4y-4=0$$

then the length of the perpendicular drawn from (1, 2), the centre of the circle, must be equal to 3, the radius of the circle

i.e., $$\pm\frac{4\times 1+3\times 2+k}{\sqrt{16+9}}=3.$$

or $$\pm\frac{4+6+k}{5}=3$$

or $$10+k=\pm 15$$

This gives $$k=+5,\ -25$$

Hence (1) gives

$$4x+3y+5=0 \quad \text{and}$$

$$4x+3y-25=0$$

which are the equations of the two tangents.

Exercise 13

Find the equation of the tangent to the circle :—

1. $4x^2+4y^2=25$ at $(-\frac{3}{2}, -2)$.
2. $2x^2+2y^2+3x-4y+1=0$ at $(-1, 2)$.
3. $x^2+y^2=10$ at the point whose abscissa is 1.
4. $x^2+y^2+2gx+2fy+c=0$ at $(0, 0)$.
5. $x^2+y^2-4x+2y+3=0$ at $(1, -2)$.

Find the equation of normal to the circle :

6. $x^2+y^2=169$ at $(5, -12)$.
7. $x^2+y^2-4x+2y+3=0$ at $(1, -2)$.
8. $4x^2+4y^2=25$ at $(-\frac{3}{2}, -2)$.
9. $2x^2+2y^2+3x-4y+1=0$ at $(-1, 2)$.
10. $x^2+y^2=\dfrac{a^2b^2}{a^2+b^2}$ at $\left(\dfrac{ab^2}{a^2+b^2}, \dfrac{a^2b}{a^2+b^2}\right)$.

Find the points of intersection of :—

11. the line $x-2y+10=0$ and the circle $x^2+y^2=25$.

12. the line $y=mx+a$ and the circle $x^2+y^2=a^2$.

13. the line $3x+4y+7=0$ and the circle

$$x^2+y^2-4x-6y-12=0.$$

14. Show that the line $3x+4y+20=0$ touches the circle $x^2+y^2=16$. Find also the point of contact. (*P.U.*)

15. Prove that $x=7$ and $y=8$ touch the circle $x^2+y^2-4x-6y-12=0$ and find the point of contact.

16. Find the condition that the line $Ax+By+C=0$ may touch the circle $x^2+y^2=a^2$.

Find the point of contact, if this condition is satisfied.

17. Find the condition that $Ax+By+C=0$ may touch the circle $(x-a)^2+(y-b)^2=c^2$.

18. Find the length of the chord joining the points in which the straight line

$$\frac{x}{a}+\frac{y}{b}=1$$

meets the circle

$$x^2+y^2=r^2.$$

19. Find the length and middle point of the chord intercepted by the circle $x^2+y^2=a^2$ on the line $y=px+q$.

20. Find the co-ordinates of the points where the line $y=2x+1$ cuts the circle $x^2+y^2=2$. Also find the length of the chord intercepted.

21. For what value of k will the line $4x+3y+k=0$ touch the circle $2x^2+2y^2-5x$.

22. Find the equation of the circle which has its centre at the point $(1, -2)$ and touches the line $x+y+5=0$.

23. Find the equations of the tangents to the circle $x^2+y^2=4$ which are inclined at an angle of 60° to the axis of x.

24. Find the equations of the tangents to the circle

$$x^2+y^2=4$$

(*i*) which are parallel to the line

$$x+2y+3=0\,;$$

(*ii*) which are parallel to the axis of y.

25. Find the equations of the tangents to the circle $x^2+y^2-2x-4y-4=0$ which are parallel to the line

$$3x-4y-1=0$$

26. Find the equations of the tangents to the circle

(*i*) $x^2+y^2=9$ which are perpendicular to $x-y-1=0$

(*ii*) $x^2+y^2-2x-4y-4=0$ which are perpendicular to the line $3x-4y-1=0$.

CHAPTER X

A CIRCLE AND A POINT

10.1. Position of a point with regard to a circle

To show that the point (x', y') *lies outside, on, or inside the circle*

$$x^2+y^2+2gx+2fy+c=0$$

according as the quantity

$x'^2+y'^2+2gx'+2fy'+c$ *is positive, zero, or negative.*

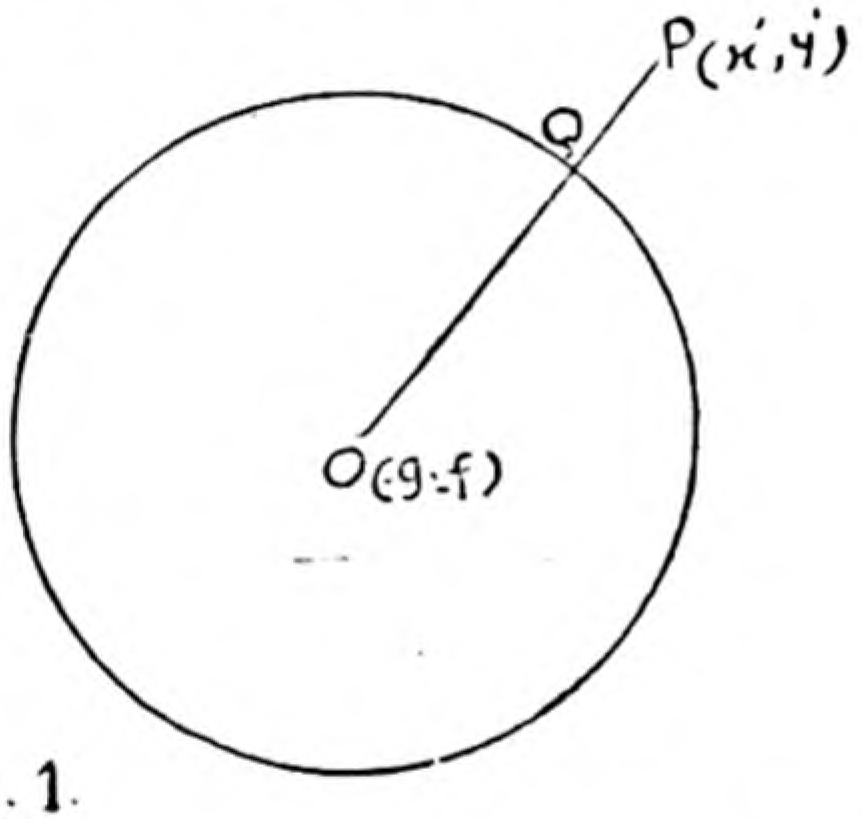


Fig. 1.

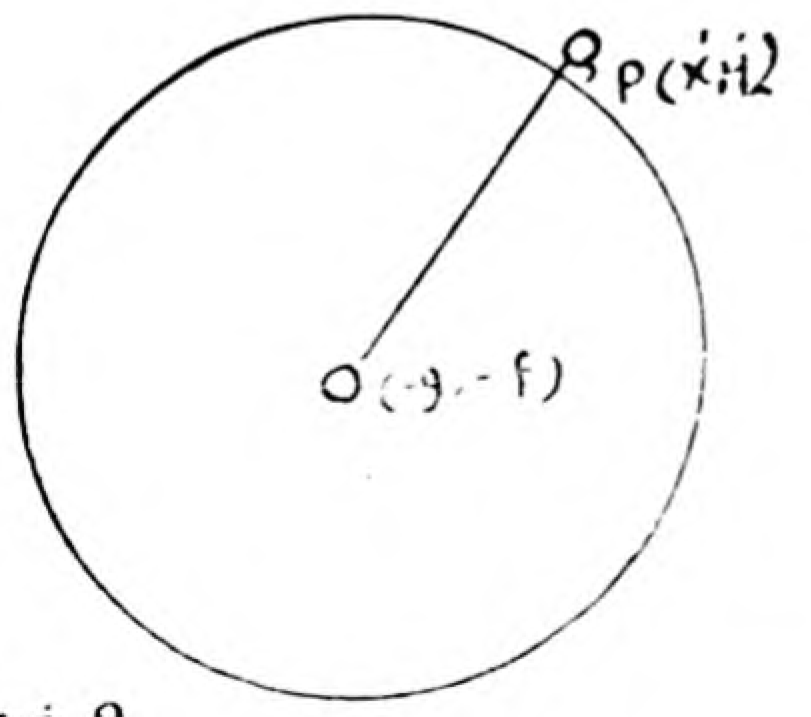


Fig 2.

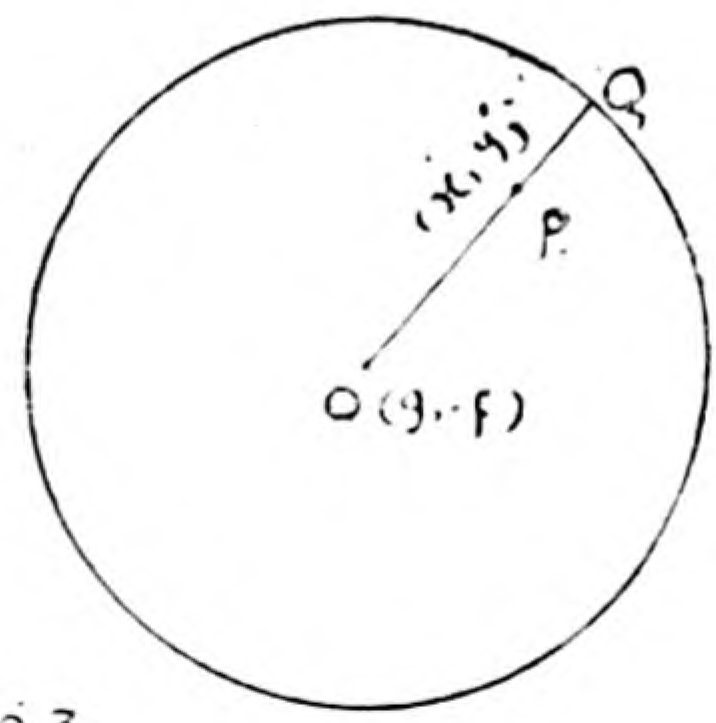


Fig 3.

Let O be the centre of the circle and P(x', y') the given point.

Case I. If P lies outside the circle, as in fig. (*i*) then OP is greater than OQ, *i.e.*, the distance between the centre and the given point is greater than the radius of the circle.

i.e., $\sqrt{(x'+g)^2+(y'+f)^2} > \sqrt{g^2+f^2-c}$

or $(x'+g)^2+(y'+f)^2 > g^2+f^2-c$

which gives

$$x'^2+y'^2+2gx'+2fy'+c > 0$$

Case II. If P lies on the circle, as in fig. (*ii*) then OP is equal to OQ

i.e., the distance between the centre and the given point is equal to the radius of the circle.

i.e., $\sqrt{(x'+g)^2+(y'+f)^2} = \sqrt{g^2+f^2-c}$

or $(x'+g)^2+(y'+f)^2 = g^2+f^2-c$

which gives

$$\mathbf{x'^2+y'^2+2gx'+2fy'+c=0}$$

Case III. If P lies inside the circle, as in fig. (*iii*), then OP is less than OQ.

i.e., the distance between the centre and the given point is less than the radius of the circle.

i.e., $\sqrt{(x'+g)^2+(y'+f)^2} < \sqrt{g^2+f^2-c}$

or $(x'+g)^2+(y'+f)^2 < g^2+f^2-c$

which gives

$$\mathbf{x'^2+y'^2+2gx'+2fy'+c<0}$$

Hence the point P(x', y') lies outside, on, or inside the circle $x^2+y^2+2gx+2fy+c=0$ *according as the quantity*

$$\mathbf{x'^2+y'^2+2gx'+2fy'+c} \gtreqless 0$$

10.2. Two tangents from a point

To show that two tangents can be drawn to a circle from a point, and these tangents will be real and distinct, coincident, or imaginary, according as the point is outside, on, or inside the circle.

In Article 9.10, we have shown that

$$y=mx+a\sqrt{1+m^2}$$

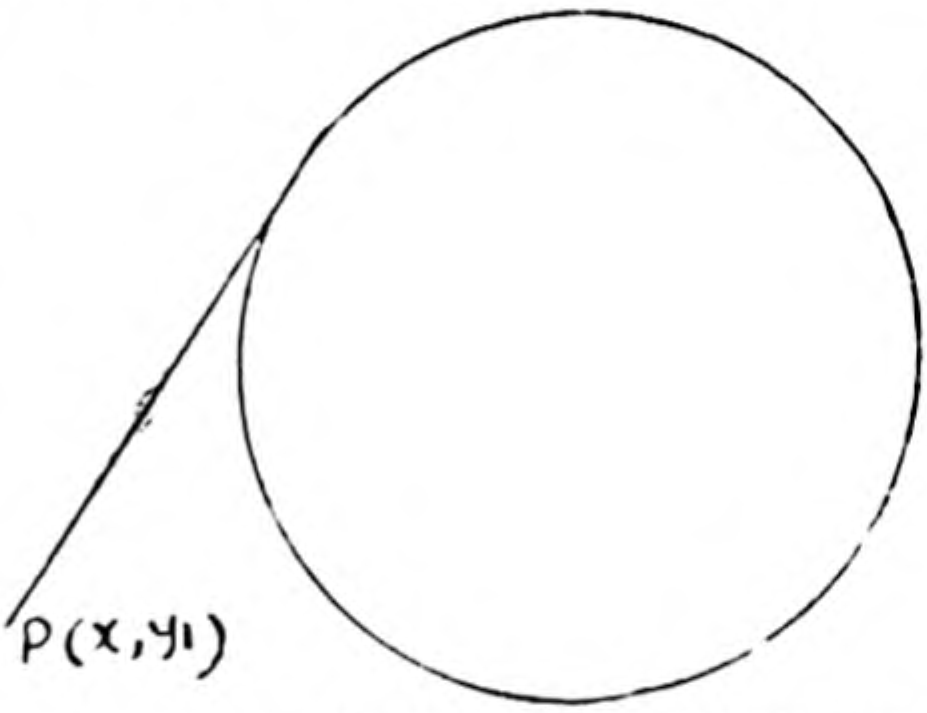
is the tangent to the circle $x^2+y^2=a^2$ *for all values of* m.

Let it be drawn from $P(x_1, y_1)$

$\therefore \quad y_1=mx_1+a\sqrt{1+m^2}$

or $\quad (y_1-mx_1)^2=a^2(1+m^2)$

or $\quad m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-a^2)=0 \ldots(1)$



Now (1) is a quadratic in m giving two values of m. These two values of m will represent two slopes of the two tangents that can be drawn to the circle $x^2+y^2=a^2$ from the point $P(x_1, y_1)$.

Hence two tangents can be drawn to a circle from a given point.

Again, these two tangents will be real and distinct, equal, or imaginary, according as the roots of (1) are real and different, coincident, or imaginary.

i e., the discriminant of (1) *is* $+ve$, *zero, or* $-ve$

i.e., $\quad 4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)\gtreqless 0$

or, $\quad x_1^2y_1^2-(x_1^2y_1^2-x_1^2a^2-a^2y'^2+a^4)\gtreqless 0$

or, $\quad x_1^2a^2+a^2y_1^2-a^4 \gtreqless 0$

or, $\quad x_1^2+y_1^2-a^2 \gtreqless 0$

i.e., the point (x_1, y_1) *lies outside, on, or inside the circle* $x^2+y^2=a^2$. (Art. 10.1)

Director Circle

10.3. *To find the locus of the point of intersection of two perpendicular tangents to a circle.*

We know that $y=mx+a\sqrt{1+m^2}$ is the tangent to the circle $x^2+y^2=a^2$ *for all values of* m. Let it be drawn from the point (x_1, y_1).

$$\therefore \quad y_1=mx_1+a\sqrt{1+m^2}$$

or $$(y_1-mx_1)^2=a^2(1+m^2)$$

or $$m^2(x_1^2-a^2)-2x_1y_1m+(y_1^2-a^2)=0 \quad \ldots(1)$$

Now (1) gives two slopes of the two tangents that can be drawn to the circle $x^2+y^2=a^2$ from the point (x_1, y_1).

These two tangents will be perpendicular to each other if the product of the two values of m (*i.e.*, of m_1 and m_2) is equal to -1.

Now from (1), $m_1 m_2=\dfrac{y_1^2-a^2}{x_1^2-a^2}$.

$$\therefore \quad m_1m_2=-1 \quad \text{gives}$$

$$\frac{y_1^2-a^2}{x_1^2-a^2}=-1$$

or $$y_1^2-a^2=-x_1^2+a^2$$

or $$x_1^2+y_1^2=2a^2$$

Hence the locus of the point (x_1, y_1) *is*

$$\mathbf{x^2+y^2=2a^2}$$

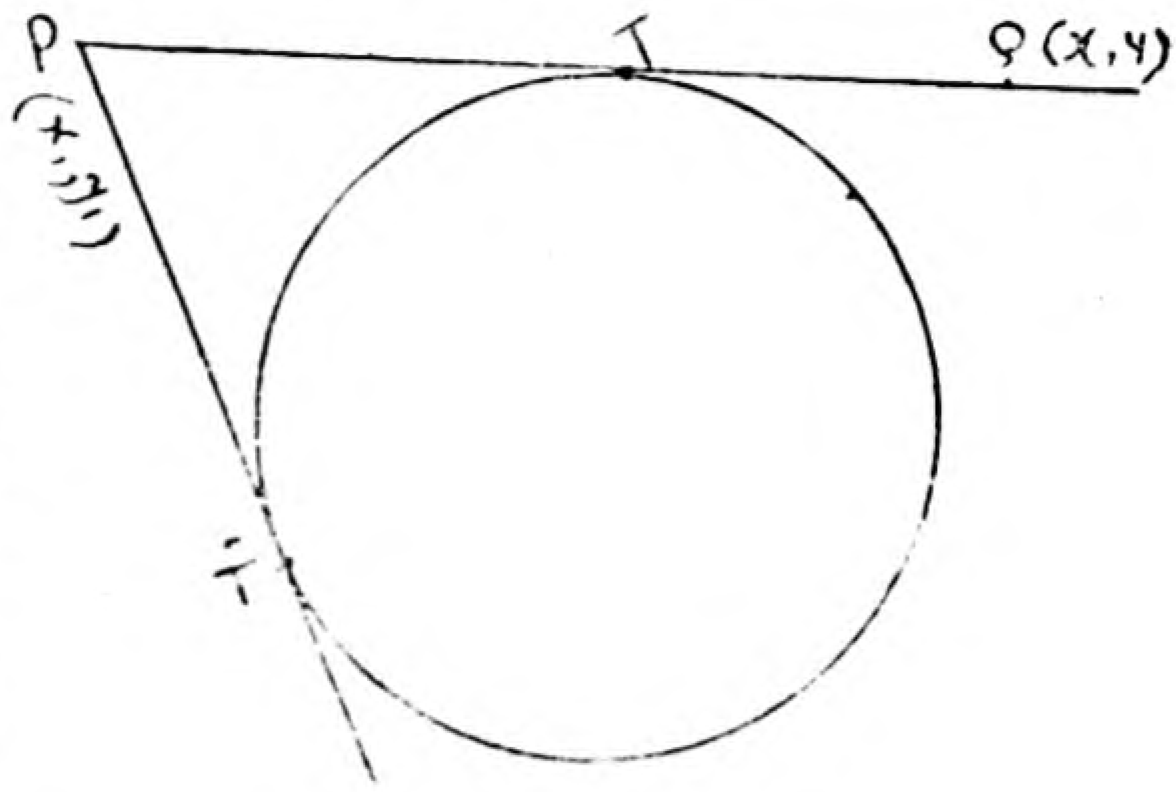
which is a circle called the *Director Circle.*

Note : *To find the locus, we change* x_1 *to* x *and* y_1 *to* y *in the resulting equation.*

Equation of the pair of tangents

10.4. *To find the equation of the pair of tangents drawn from the point* (x_1, y_1) *to the circle* $x^2+y^2=a^2$.

Let $P(x_1, y_1)$ be the point from which the two tangents are drawn, and let PT and PT′ be these two tangents drawn from this point. Let Q (x, y) be any other point on the tangent PT.



Suppose that T divides PQ in the ratio $k : 1$.

Then the co-ordinates of T are

$$\left(\frac{kx+x_1}{k+1},\ \frac{ky+y_1}{k+1}\right)$$

Now T lies on the circle

$$\therefore \qquad \left(\frac{kx+x_1}{k+1}\right)^2+\left(\frac{ky+y_1}{k+1}\right)^2=a^2$$

or $$(kx+x_1)^2+(ky+y_1)^2=a^2(k+1)^2$$

or $$k^2(x^2+y^2-a^2)+2k(xx_1+yy_1-a^2)$$
$$+(x_1^2+y_1^2-a^2)=0 \qquad \ldots(1)$$

$\because$ PQ touches the circle, (1) must have two equal values of k

$$\therefore \qquad (xx_1+yy_1-a^2)^2=4(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)$$

or $$\mathbf{(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)=(xx_1+yy_1-a^2)^2}$$

which gives the required equation of the pair of tangents.

Note : *The equation of the pair of tangents drawn from* (x_1, y_1) *to the circle* $x^2+y^2=a^2$ *can be briefly written as under :*

$$\mathbf{SS_1=T_1^2}$$

where $\quad S = x^2+y^2-a^2$

$\quad S_1 = x_1^2+y_1^2-a^2$

and $\quad T_1 = xx_1+yy_1-a^2$

Explanation of the above notation :—

(*i*) $S=0$ is the equation of the circle

$$x^2+y^2-a^2=0$$

(*ii*) S_1 is the result after substituting the co-ordinates (x_1, y_1) in S, and $T_1=0$ is equation of the tangent to the circle $S=0$ at (x_1, y_1).

Illustration :

Find the equation of the pair of tangents drawn from the point (2, 2) to the circle $x^2+y^2=1$

Sol. Here $\quad S=x^2+y^2-1$

$\quad S_1=(2)^2+(2)^2-1=4+4-1=7$

$\quad T_1=2x+2y-1$

$\therefore$ the required equation is

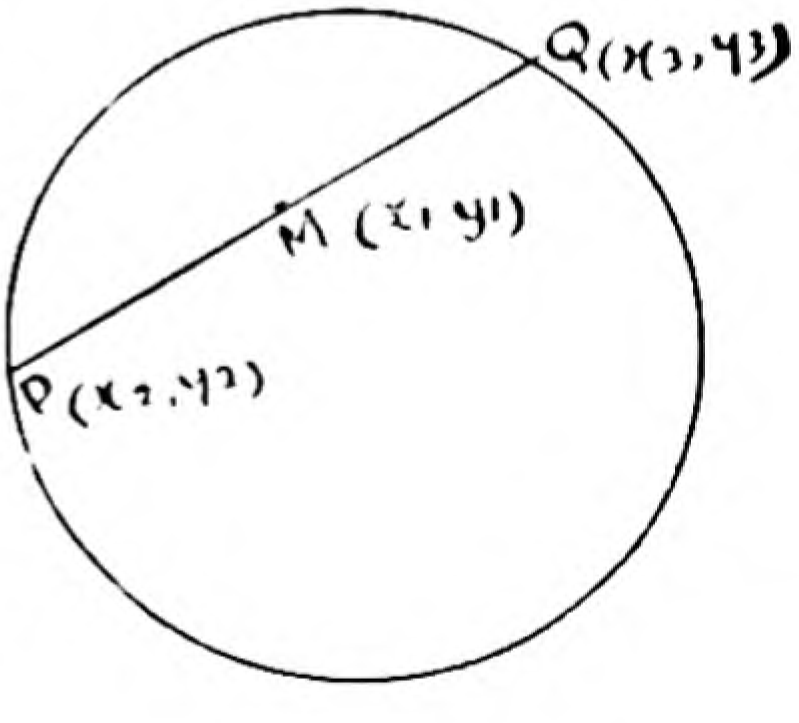
$$7(x^2+y^2-1)=(2x+2y-1)^2 \ldots\ldots (SS_1=T_1^2)$$

or $$7x^2+7y^2-7=4x^2+4y^2+1+8xy-4y-4x$$

or $$3x^2+3y^2-8xy+4x+4y-8=0$$

Equation of a chord in terms of its mid-point

10.5. *To find the equation of a chord of the circle* $x^2+y^2=a^2$ *whose mid-point is* (x_1, y_1).



Let $P(x_2, y_2)$ and $Q(x_3, y_3)$ be the extremities of the chord PQ and $M(x_1, y_1)$ its mid-point. Then

$$\frac{x_2+x_3}{2}=x_1 \quad \text{or} \quad x_2+x_3=2x_1$$

and $$\frac{y_2+y_3}{2}=y_1 \quad \text{or} \quad y_2+y_3=2y_1$$

Now $P(x_2, y_2)$ and $Q(x_3, y_3)$ lies on the circle $\quad x^2+y^2=a^2$

$$\therefore \qquad x_2^2+y_2^2=a^2 \qquad \ldots(1)$$

$$x_3^2+y_3^2=a^2 \qquad \ldots(2)$$

Subtracting (1) from (2), we get

$$(x_3^2-x_2^2)+(y_3^2-y_2^2)=0$$

or $$(x_3+x_2)(x_3-x_2)=-(y_3+y_2)(y_3-y_2)$$

or $$\frac{y_3-y_2}{x_3-x_2}=-\frac{x_3+x_2}{y_3+y_2}=\text{slope of PQ}$$

(*Please note this step*)

Now the equation of the chord PQ [through (x_1, y_1)] is given by

$$y-y_1=\frac{y_3-y_2}{x_3-x_2}(x-x_1)$$

$$\left(\because \text{slope of PQ}=\frac{y_3-y_2}{x_3-x_2}\right)$$

or $$y-y_1=-\frac{x_3+x_2}{y_3+y_2}(x-x_1)$$

$$\left(\because \frac{y_3-y_2}{x_3-x_2}=-\frac{x_3+x_2}{y_3+y_2}\right)$$

or $$y-y_1=-\frac{2x_1}{2y_1}(x-x_1)$$

$$\left(\because \begin{matrix} x_2+x_3=2x_1 \\ y_2+y_3=2y_1 \end{matrix}\right)$$

or $$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

or $$yy_1-y_1^2=-xx_1+x_1^2$$

or $$\mathbf{xx_1+yy_1=x_1^2+y_1^2} \qquad \ldots(3)$$

which is the required equation.

Note : (Important)

Equation (3) can be put as

$$xx_1+yy_1-a^2=x_1^2+y_1^2-a^2$$

or $$x_1^2+y_1^2-a^2=xx_1+yy_1-a^2$$

or $$\mathbf{S_1=T_1}$$

which is the **abridged form** of the chord.

Explanation :

(*i*) S=0 is the equation of the circle

$$x^2+y^2-a^2=0$$

(*ii*) S_1 is the result of substituting the co-ordinates (x_1, y_1) in S, and

(*iii*) $T_1=0$ is the equation of the tangent to the circle S=0 at (x_1, y_1).

Hence the equation of the chord of the circle $x^2+y^2=a^2$ in terms of its mid-point (x_1, y_1) can be briefly put as

$$S_1=T_1$$

where $$S_1=x_1^2+y_1^2-a^2$$

and $$T_1=xx_1+yy_1-a^2$$

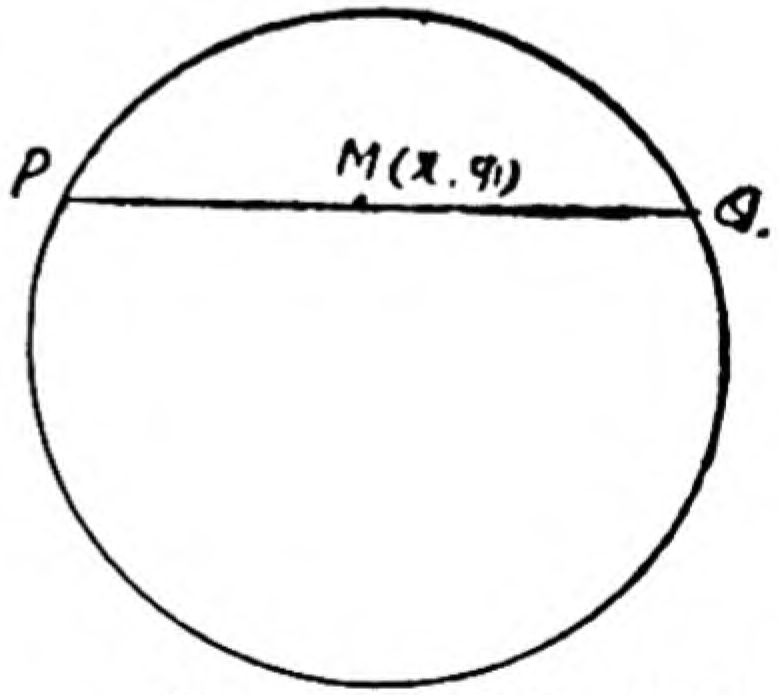
The Diameter

10.6. *To find the locus of the middle points of a system of parallel chords of a circle.*

Let the equation of the circle be

$$x^2+y^2-a^2=0$$

Let m be the slope of the parallel chords, and $M(x_1, a_1)$ be the mid-point of one of such parallel chords PQ. Then the equation of PQ in terms of the mid-point is



$$xx_1+yy_1=x_1^2+y_1^2 \qquad ...(1) \qquad \text{(Article 10.5)}$$

$$\text{Slope of (1)} \quad =-\frac{x_1}{y_1}=m \qquad ...\text{(supposed above)}$$

This gives $y_1=-\frac{1}{m}x_1$

∴ the locus of (x_1, y_1) is

$$\mathbf{y=-\frac{1}{m}x}$$

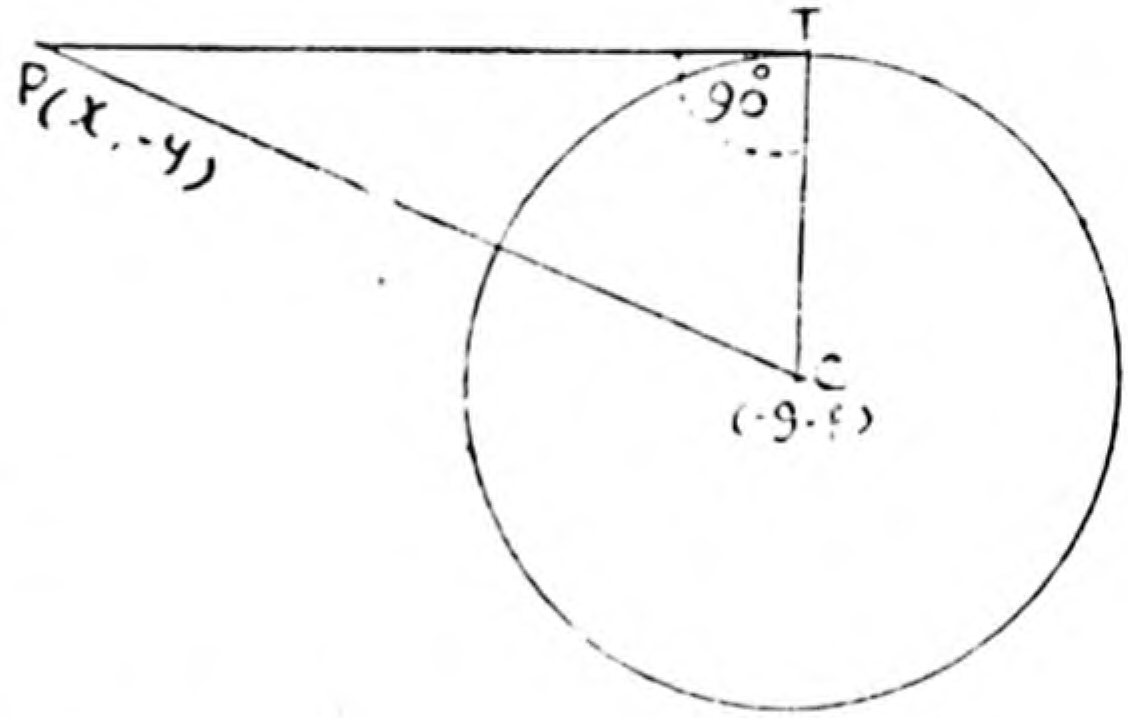
which is a st. line *passing through the origin, i.e.*, through the centre of the circle, and is called the **Diameter** of the circle.

Hence, the locus of the middle points of a system of parallel chords of a circle is called a **Diameter.**

Length of the Tangent

10.7. *To find the length of the tangent from the point* (x_1, y_1) *to the circle* $x^2+y^2+2gx+2fy+c=0$.

Let $P(x_1, y_1)$ be the point, PT the tangent from P to the circle, T the point of contact, and $C(-g, -f)$ the centre of the circle.



Join CT and CP. Then from the right-angled △CTP

$$PT^2=PC^2-CT^2 \quad ...(1)$$

Now $PC^2=[\{x_1-(-g)\}^2+\{y_1-(-f)\}^2]$ (Distance formula)

$$=(x_1+g)^2+(y_1+f)^2$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+g^2+f^2$$

and $CT^2=(\text{radius})^2=g^2+f^2-c$

∴ (1) gives

$$PT^2=(x_1^2+y_1^2+2gx_1+2fy_1+g^2+f^2)-(g^2+f^2-c)$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c$$

Hence $\mathbf{PT=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}}$

Note : Working rule to find the length of the tangent from a given point to a given circle.

In the L.H.S. of the equation of the circle (after making the coefficients of x^2 and y^2 each equal to unity, if necessary), substitute the co-ordinates of the point, and take the square root of the result.

This will give the length of the tangent.

Illustration : Find the length of the tangent drawn from the point $(1, -1)$ to the circle

$$2x^2+2y^2-6x-3y+17=0.$$

Sol. The equation of the circle can be put as

$$x^2+y^2-3x-\tfrac{3}{2}y+\tfrac{17}{2}=0 \qquad \textit{(Please note this)}$$

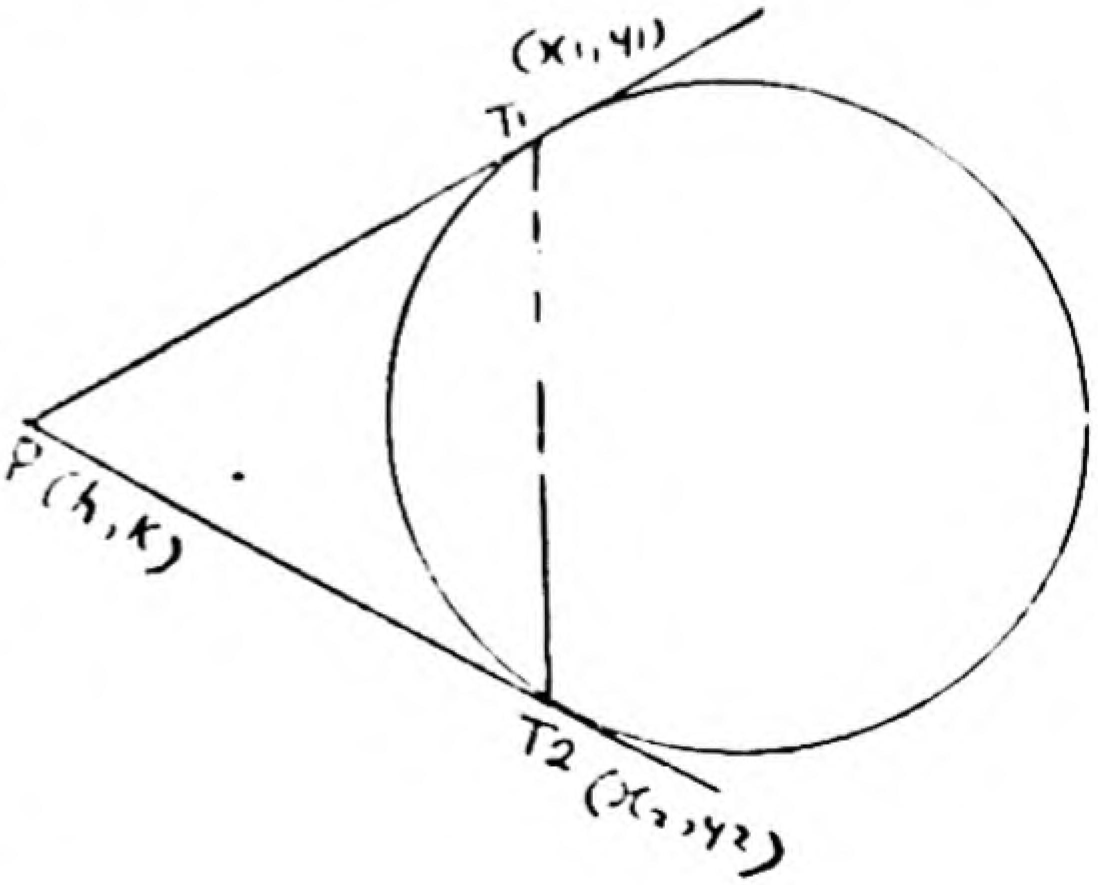
$\therefore$ the length of the tangent from $(1, -1)$

$$=\sqrt{(1)^2+(-1)^2-3(1)-\tfrac{3}{2}(-1)+\tfrac{17}{2}}=\sqrt{9}=3$$

Chord of Contact

10.8. Def. *If from any point P outside a circle two tangents PT_1 and PT_2 be drawn to the circle, the straight line T_1T_2 joining the points of contact is called the* **Chord of Contact** *of tangents from P.*

To find the equation of the chord of contact of tangents drawn to the circle $x^2+y^2=a^2$ from the external point (h, k)).



Let P be the point (h, k) and T_1 and T_2 the points of contact of the two tangents drawn from P to the circle. Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of T_1 and T_2 respectively.

The tangent at T_1 is

$$xx_1+yy_1=a^2 \quad ...(1)$$

and that at T_2 is

$$xx_2+yy_2=a_2 \quad ...(2)$$

Since both these

tangents pass through $P(h, k)$, we have

$$hx_1+ky_1=a^2 \ldots (3)$$

and $$hx_2+ky_2=a^2 \ldots (4)$$

Now, (3) and (4) show that a straight line

$$\mathbf{hx+ky=a^2}$$

passes through (x_1, y_1) and (x_2, y_2), and this, therefore, represents the equation of the *chord of contact.*

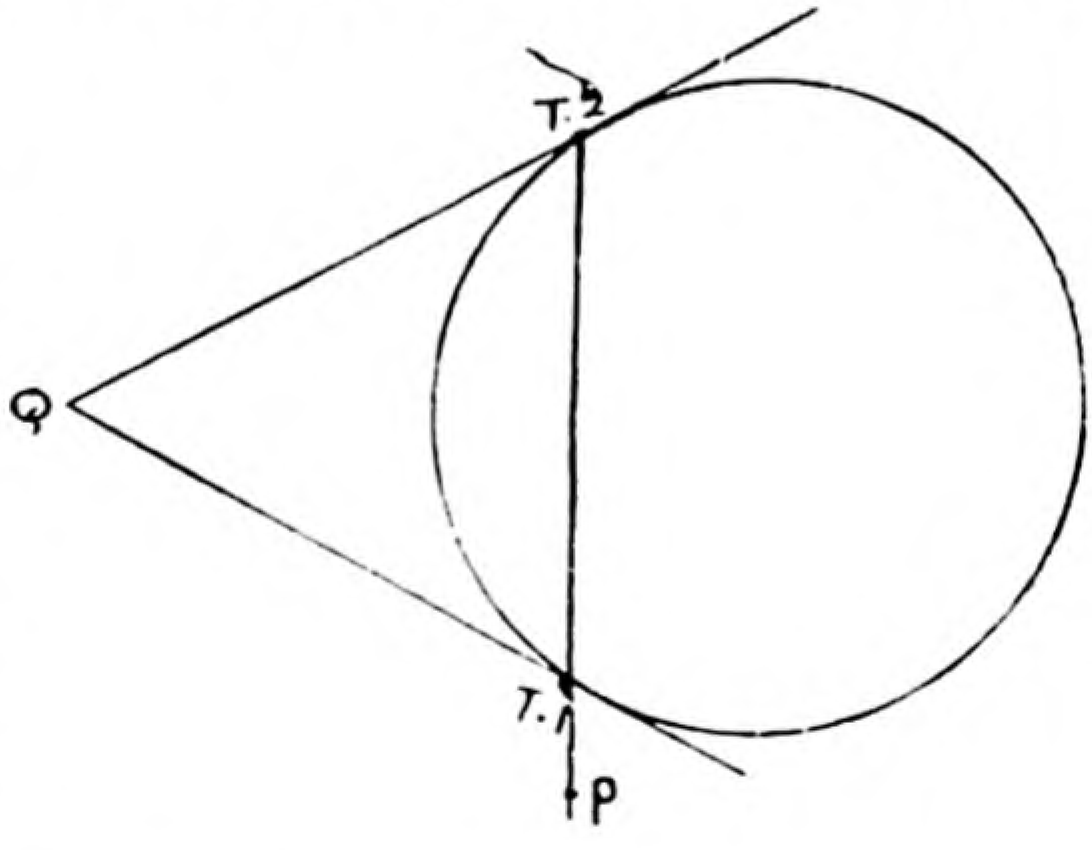
Note : *Equation of the chord of contact of the tangents drawn from* (h, k) *to the circle* $x^2+y^2+2gx+2fy+c=0$ *is* $hx+ky+g(x+h)+f(y+k)+c=0$.

Pole and Polar

10.9. Def. *The locus of the point of intersection of tangents drawn to a circle at the extremities of any chord passing through a fixed point is called the* **Polar** *of the point, and the point is called the* **Pole** *of the line.*

Explanation :

If from a point P, a chord is drawn to cut the circle in T_1 and T_2, and tangents drawn at T_1 and T_2 meet at Q, then the locus of Q is called the **Polar** of the point P with respect to the circle. The point P is called the **Pole** with respect to the circle.



In the following Article, it will be shown that this locus is a st. line.

10.10. Equation of the Polar of a point. *To find the polar of the point* (h, k) *with respect to the circle* $x^2+y^2=a^2$.

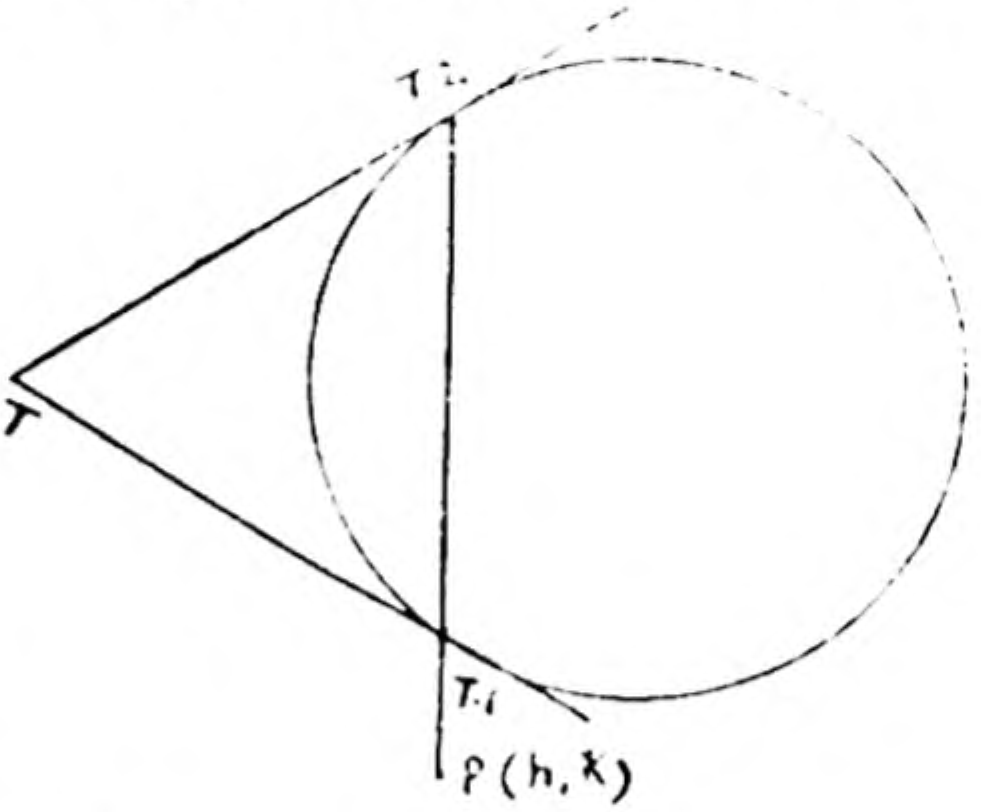
Let P be the point (h, k). PT_1T_2 is drawn to cut the circle in T_1 and T_2. Let the tangents drawn at T_1 and T_2 meet at $T(x_1, y_1)$. Then we have to find the *locus* of T.

Now equation of the chord of contact of $T\ (x_1, y_1)$ is

$$xx_1 + yy_1 = a^2 \ldots$$

(Article 10·8)

This is the equation of T_1T_2, which passes through $P\ (h, k)$ •



$$\therefore \qquad hx_1 + ky_1 = a^2$$

Hence the locus of $T(x_1, y_1)$ *is the line*

$$\mathbf{hx + ky = a^2}$$

which gives the equation of the *Polar* of the point $P(h, k)$

Note : (1) *The equation of the Polar of the point* (h, k) *with respect to the circle* $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$hx + ky + g(x + h) + f(y + k) + c = 0.$$

(2) *The student should carefully understand the difference between the chord of contact and the Polar. The equation of the tangent at* (h, k), *the equation of the chord of contact of the tangents drawn from the point* (h, k). *and the equation of the polar of the point* (h, k) *are all* **identical** *in form, but they are altogether different in significance.*

(3) (*i*) *If the point* $P(h, k)$ *is outside the circle, the two tangents are real, and hence their points of contact are real. Therefore its polar is the same as the chord of contact of the tangents from it (see fig.* (*i*) *below)*

(*ii*) *If the paint* $P(h, k)$ *is on the circle, its polar is the same as tangent at P (see fig.* (*ii*) *below)*

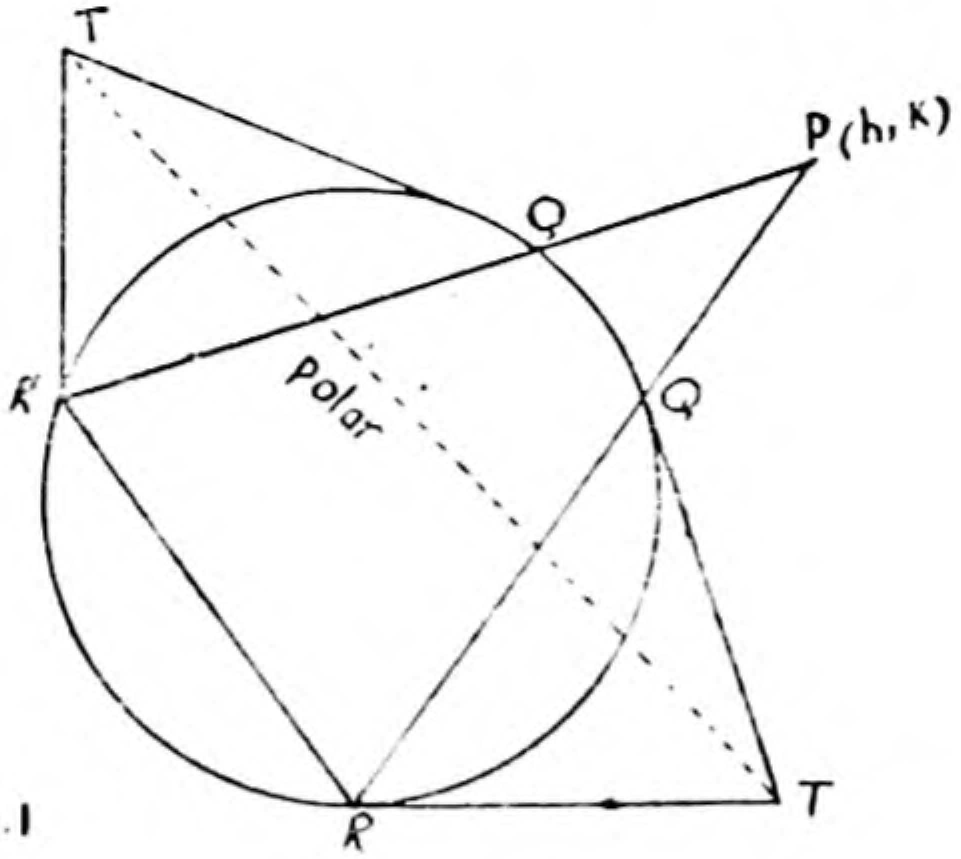


Fig. 1

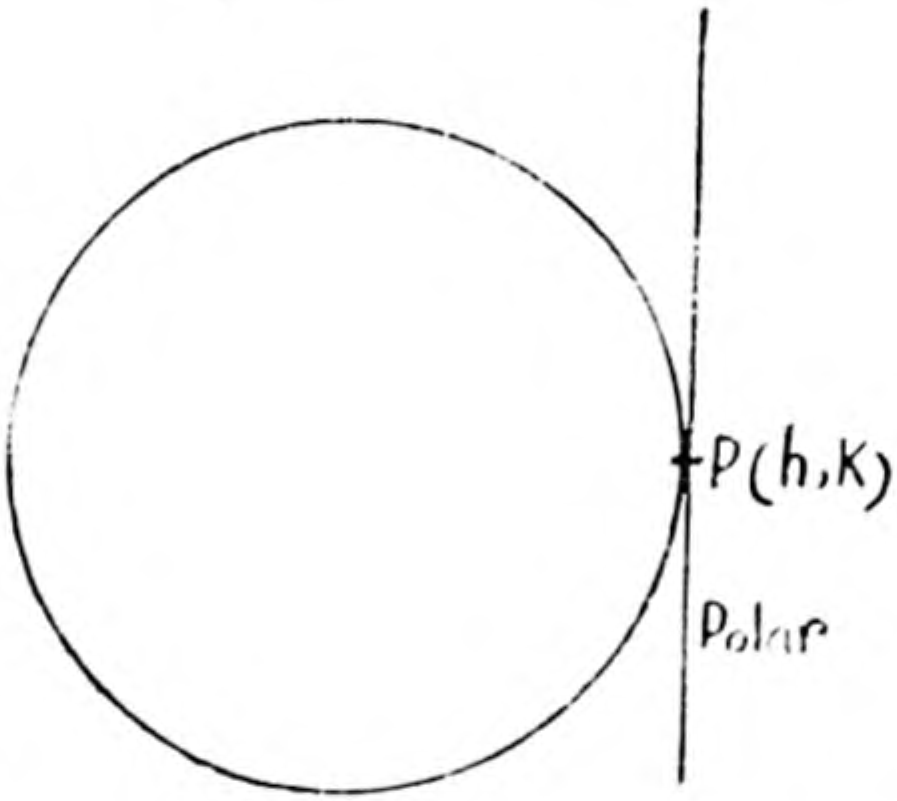


Fig. 2.

(4) *The words "Pole with respect to the circle" should be noted. We say "Pole with respect to (briefly written as w.r.t.) the circle" and not "Pole of the circle".*

10.11. Pole of a Line.

To find the pole of the line $lx+my+n=0$ w.r.t. the circle $x^2+y^2=a^2$.

Let $P(h, k)$ be the pole of the given line.

Now the polar of $P(h, k)$ with respect to the circle

$x^2+y^2=a^2$ is

$$hx+ky-a^2=0 \qquad \ldots(1) \text{ (Art. 10.10)}$$

This is the same as

$$lx+my+n=0 \qquad \ldots(2)$$

$\therefore$ comparing coefficients of (1) and (2) we get

$$\frac{h}{l}=\frac{k}{m}=\frac{-a^2}{n}$$

which gives $h=\dfrac{-a^2l}{n}$, $k=\dfrac{-a^2m}{n}$.

Hence the pole is

$$\left(\frac{-\mathbf{a^2l}}{\mathbf{n}}, \frac{-\mathbf{a^2m}}{\mathbf{n}}\right).$$

10.12. Properties of Pole and Polar

(*a*) **Conjugate points.** *If the polar of a point P w.r.t. a circle passes through Q, then the Polar of Q will pass through P.*

Let the circle be

$$x^2+y^2=a^2$$

and the co-ordinates of P and Q be (x_1, y_1) and (x_2, y_2) respectively.

The equation of the *polar* of P w.r.t. the circle is

$$xx_1+yy_1=a^2 \qquad \text{(Art. 10.10)}$$

If this passes through $Q(x_2, y_2)$, we have

$$\mathbf{x_2x_1+y_2y_1=a^2} \qquad \ldots(1)$$

Now (*1*) *shows that the equation*

$$x_2x+y_2y=a^2$$

passes through $P(x_1, y_1)$. *But* $x_2x+y_2y=a^2$ (*or* $xx_2+yy_2=a^2$) *is the polar of* Q *with respect to the circle* $x^2+y^2=a^2$. *Hence the polar of Q passes through P.*

Def. *Two points such that the polar of each passes through the other are called conjugate points.*

(1) is the condition for the points (x_1, y_1) and (x_2, y_2) to be conjugate *w.r.t.* the circle $x^2+y^2=a^2$.

(*b*) **Conjugate lines.** *If the pole of a line p w.r.t. a circle lies on a line q, then the pole of q lies on p.*

Let the circle be $x^2+y^2=a^2$ ...(1)

Let the equations of p and q be respectively

$$l_1x+m_1y+n_1=0 \quad ...(2)$$

$$l_2x+m_2y+n_2=0 \quad ...(3)$$

Then the pole of (2) *w.r.t.* the circle

$$x^2+y^2=a^2 \text{ is } \left(\frac{-a^2l_1}{n_1}, \frac{-a^2m_1}{n_1}\right). \qquad \text{(Article 10.11)}$$

If this lies on (3), we have

$$l_2\left(-\frac{a^2l_1}{n_1}\right)+m_2\left(-\frac{a^2m_1}{n_1}\right)+n_2=0$$

or $$\mathbf{a^2(l_1l_2+m_1m_2)=n_1n_2} \quad ...(4)$$

The symmetry of the result shows that this is also the condition for the pole of q to lie on p.

Def. *Two lines such that the pole of each lies on the other are called conjugate lines.*

(4) is the condition for the lines (2) and (3) to be conjugate *w.r.t.* the circle $x^2+y^2=a^2$.

(*c*) *If OP meets the polar of P with respect to a circle in Q, then*

(*i*) *OP is perpendicular to the polar of P.*

(*ii*) *OP.OQ=(radius)² where O is the centre of the circle.*

Let the equation of the circle be

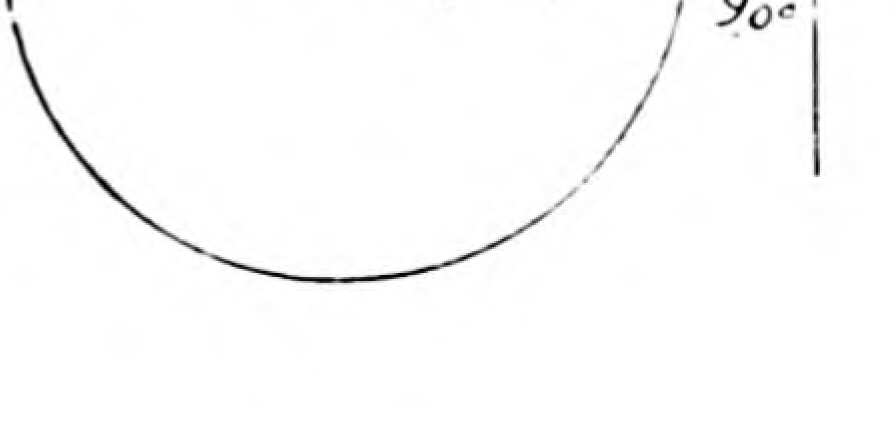
$$x^2+y^2=a^2$$

and let P be (h, k).

Then the polar of P is

$$hx+ky=a^2 \quad ...(1)$$

(Article 10.4)



Slope of (1) $=-\dfrac{h}{k}$

Also slope of $OP=\frac{k-0}{h-0}=\frac{k}{h}$

$$\left(\because \quad m=\frac{y_2-y_1}{x_2-x_1}\right)$$

$\therefore$ the product of the two slopes=

$$-\frac{h}{k}\times\frac{k}{h}=-1$$

Hence OP is $\perp$ to the polar of P

(*ii*) Again, $OP=\sqrt{h^2+k^2}$ ...(Distance formula)

and $OQ=\frac{a^2}{\sqrt{h^2+k^2}}$

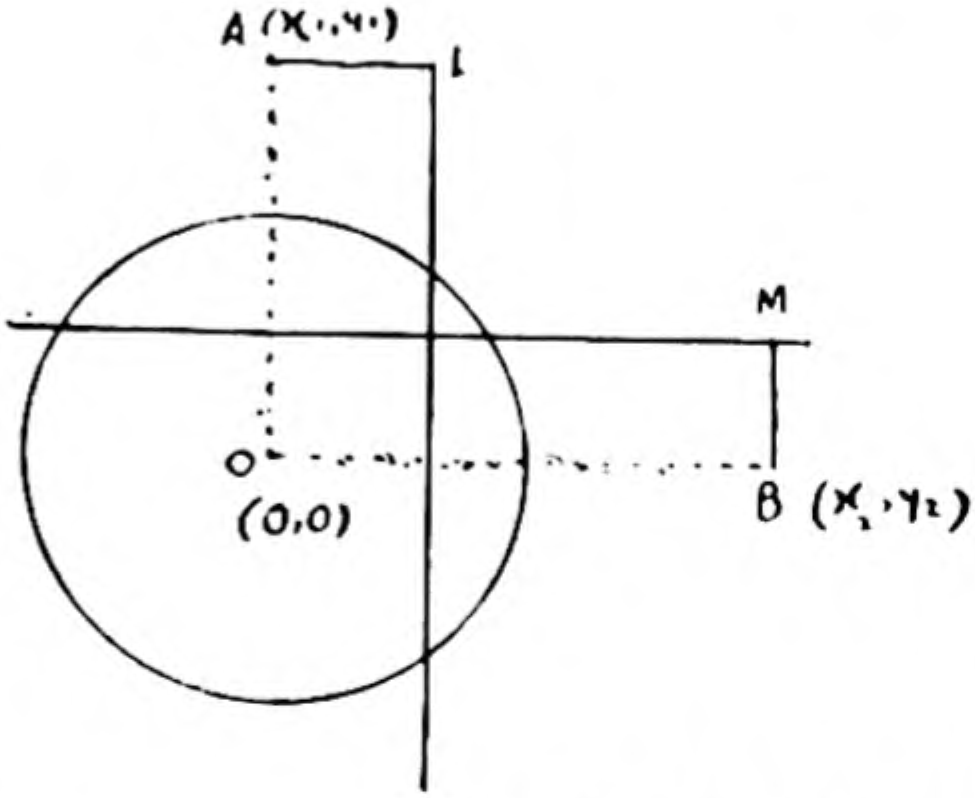
($\because$ OQ=length of $\perp$ from (h, k) on $hx+ky=a^2$)

$\therefore$ $\mathbf{OP.OQ=\sqrt{h^2+k^2}.\ \frac{a^2}{\sqrt{h^2+k^2}}=a^2=(radius)^2}$

Note : *P and Q are called the inverse points.*

10.13. Salmon's Theorem

To prove that the distances of two points from the centre of a circle are proportional to the distances of each from the polar of the other.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two points and $x^2+y^2=a^2$ be the equation of the circle.

Now the polar of $A(x_1, y_1)$ *w. r. t.* the circle is

$$xx_1+yy_1-a^2=0 \quad ...(1)$$

(Article 10.10)

Also the polar of $B(x_2, y_2)$ *w.r.t.* the same circle is

$$xx_2+yy_2-a^2=0 \quad ...(2) \text{ (Article 10.10)}$$

Draw AL and BM $\perp$s on the polars represented by (2) and (1) respectively.

$$\left.\begin{aligned}\therefore\quad AL&=\frac{x_1x_2+y_1y_2-a^2}{\sqrt{x_2^2+y_2^2}}\\ \text{and}\quad BM&=\frac{x_2x_1+y_2y_1-a^2}{\sqrt{x_1^2+y_1^2}}\end{aligned}\right\}\text{(Perpendicular formula)}$$

$$\frac{\mathbf{AL}}{\mathbf{BM}}=\frac{\sqrt{\mathbf{x}_1^2+\mathbf{y}_1^2}}{\sqrt{\mathbf{x}_2^2+\mathbf{y}_2^2}}=\frac{\mathbf{OA}}{\mathbf{OB}}$$

which proves the theorem.

Solved Examples

Ex. 1. Show that the point (1, 2) lies outside the circle

$$x^2+y^2+6x-8y+7=0.$$

Sol. Here the equation of the circle is

$$x^2+y^2+6x-8y+7=0.$$

Substituting 1 for x and 2 for y in the L.H.S., we get

$$(1)^2+(2)^2+6(1)-8(2)+7$$
$$=1+4+6-16+7=18>0$$

$\therefore$ the point (1, 2) lies outside the circle.

Ex. 2. Find the equations of the tangents drawn from the point (4, 2) to the circle $x^2+y^2=4$.

Sol. The equation of the circle is

$$x^2+y^2=4$$

Here "a"$=2$

$\therefore$ $y=mx+2\sqrt{1+m^2}$ will be tangent to the circle $x^2+y^2=4$ for all values of m.

Now this tangent passes through (4, 2).

$\therefore$ $$2=4m+2\sqrt{1+m^2}$$

or $$1=2m+\sqrt{1+m^2}$$

or $$(1-2m)^2=(1+m^2)$$

(Please note this)

or $\quad 4m^2-4m+1=1+m^2$

or $\quad 3m^2-4m=0$

or $\quad m=0, \frac{4}{3}$

$\therefore \quad y \quad mx+2\sqrt{1+m^2}$ gives

$$y=0x+2\sqrt{1+0} \text{ or } y=2 \qquad \ldots(1)$$

and $\quad y=\frac{4}{3}x+2\sqrt{1+\frac{16}{9}}$

or $\quad y=\frac{4}{3}x+\frac{10}{3}$

or $$4x-3y+10=0 \qquad \ldots(2)$$

$\therefore$ (1) and (2) are the equations of the tangent.

Ex. 3. Find the equation of the pair of tangents drawn from the point (2, 2) to the circle $x^2+y^2=1$.

Sol. Here $\quad S=x \ +y^2-1$

Substituting 2 for x and 2 for y in it, we get

$$S_1=(2)^2+(2)^2-1=4+4-1=7$$

Also $$T_1=xx_1+yy_1-1$$
$$=x\times 2+y\times 2-1=2x+2y-1$$

$\therefore \quad SS_1=T_1^2$ gives

$$(x^2+y^2-1)(7)=(2x+2y-1)^2$$

or $\quad 7x^2+7y^2-7=4x^2+4y^2+8xy^2+8xy-4y-4x+1$

or $\quad 3x^2+3y^2-8xy+4x+4y-8=0$

which gives the required equation of the pair of tangents.

Ex. 4. Find the equation to that chord of the circle $x^2+y^2=9$ which is bisected at the point $(-2, 3)$.

Sol. Here the point $(-2, 3)$ is the mid-point of the chord whose equation we are required to find. In other words, we have to find the equation of the chord in terms of its mid-point whose co-ordinates are $(-2, 3)$.

Here $$S=x^2+y^2-9 \qquad \ldots(1)$$

Substituting -2 for x and 3 for y in (1) we get

$$S_1=(-2)^2+(3)^2-9=4+9-9=4$$

Also $\quad T_1=xx_1+yy_1-9$

$$=x(-2)+y(3)-9=-2x+3y-9$$

(Substituting -2 for x_1 and 3 for y_1)

$\therefore \quad S_1=T_1$ gives

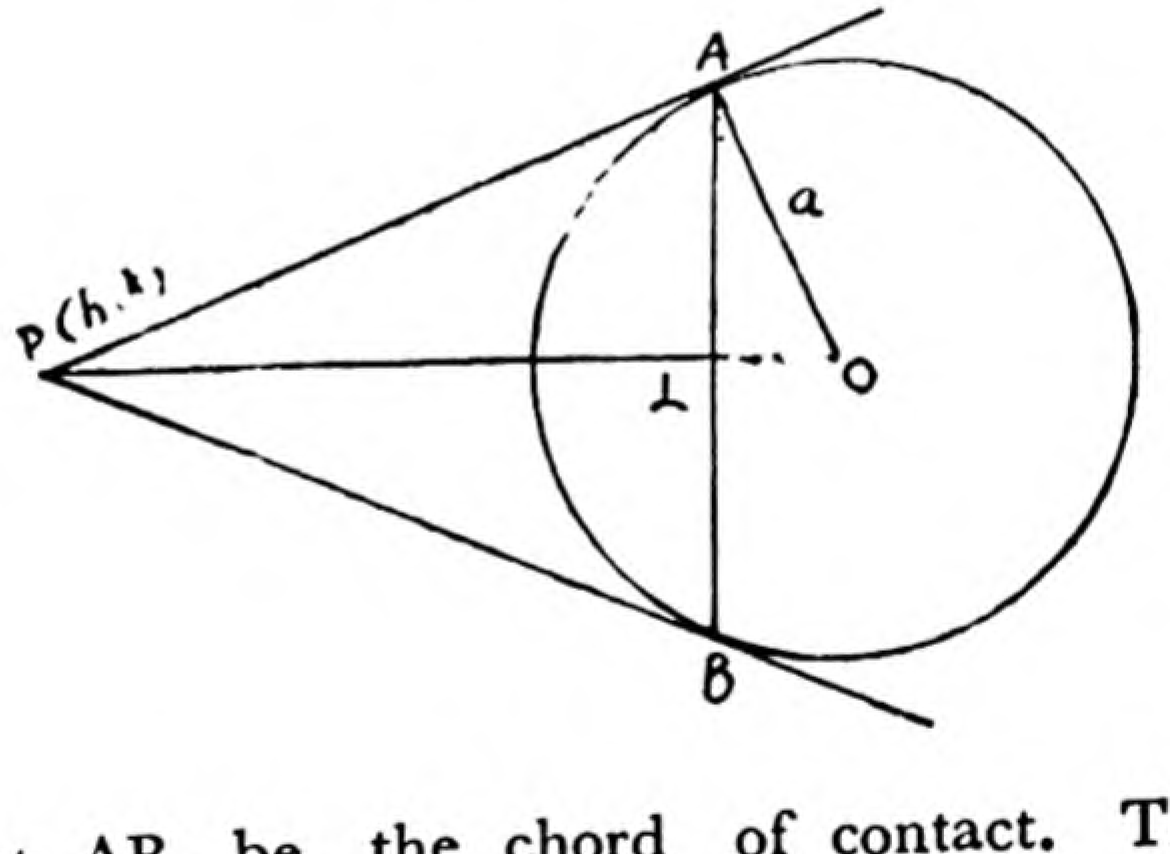
$$4=-2x+3y-9$$

or $2x-3y+13=0$

which gives the required equation of the chord.

Ex. 5. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$. Prove that the area of the triangle formed by them and the chord of contact is

$$\frac{a\,(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}$$



Sol. Let AB be the chord of contact. Therefore its equation is

$$hx+ky-a^2=0.$$

Let PL be the length of the perpendicular drawn from P(h, k) upon AB

$$\therefore \quad PL=\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}$$

Again, $AB=2AL=2\sqrt{AP^2-PL^2}$

$$=2\sqrt{(h^2+k^2-a^2)-\frac{(h+k^2-a^2)^2}{h^2+k^2}}$$

$$=2a\sqrt{\frac{h^2+k^2-a^2}{h^2+k^2}}$$

$\therefore$ the area of the triangle $PAB=\frac{1}{2}.PL.AB.$

$$=\frac{1}{2}.\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}.2a\sqrt{\frac{h^2+k^2-a^2}{h^2+k^2}}$$

$$=\frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+a^2}$$

Note : Triangle PAB is an isoscelese triangle. Therefore PL, the ⊥ on AB, will bisect AB.

Ex. 6. Prove that the polar of the point (f, g) *w.r.t.* the circle $x^2+y^2=r^2$ touches $(x-h)^2+(y-k)^2=a^2$, if

$$(fh+gk-r^2)^2=a^2(f^2+g^2)$$

Sol. The polar of (f, g) *w.r.t.* the circle

$$x^2+y^2=r^2 \text{ is}$$

$$fx+gk-r^2=0 \qquad \ldots(1)$$

It will touch the circle $(x-h)^2+(y-k)^2=a^2$ if the length of the perpendicular drawn from (h, k), the centre of the circle, upon (1) is equal to "a" the radius of the circle

$$i.e., \text{ if } \frac{fh+gk-r^2}{\sqrt{f^2+g^2}}=a$$

or

$$(fh+gk-r^2)^2=a^2(f^2+g^2)$$

Ex. 7. Find the locus of a point the chord of contact of tangents from which subtends a right angle at the centre of the circle $x^2+y^2=a^2$.

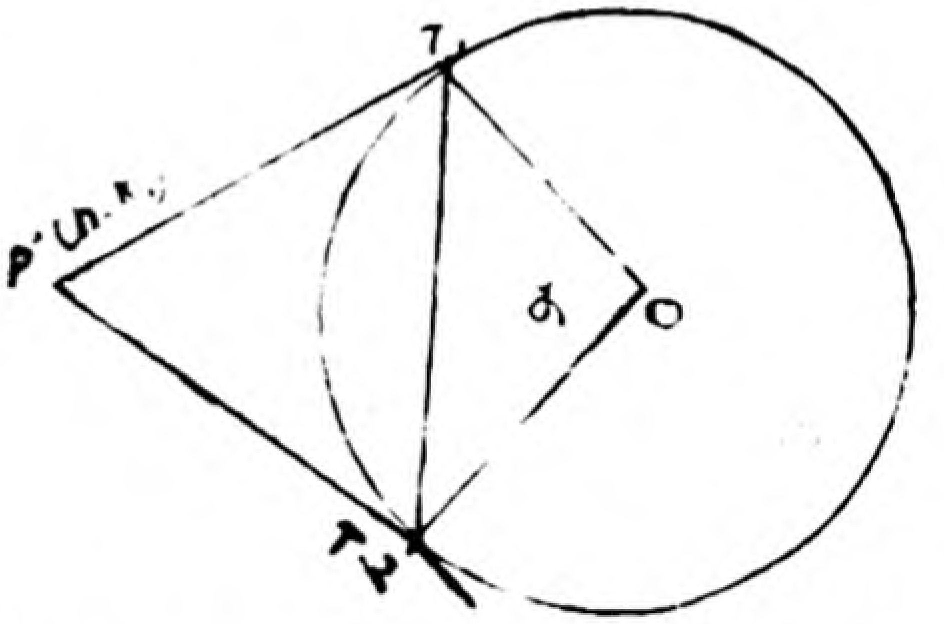
Sol. Let the point be P (h,k).

Therefore, its chord of contact *w.r.t.* the circle $x^2+y^2=a^2$ is

$$hx+ky-a^2=0$$

This can be written as

$$\frac{hx+ky}{a^2}=1 \quad \ldots(1)$$



Now the joint equation of the straight lines T_1O and T_2O (where T_1 and T_2 are the extremities of the chord of contact and O the centre of the circle) must be homogeneous of second order in x, y. This can be obtained by making the equation of the circle homogeneous with the help of (1) ...(Article 7·9)

The process is shown as under :—

$$x^2+y^2-a^2\left(\frac{hx+ky}{a^2}\right)^2=0$$

or $$x^2+y^2-\frac{h^2x^2+2hkxy+k^2y^2}{a^2}=0$$

or $$a^2x^2+a^2y^2-h^2x^2-2hkxy-k^2y^2=0$$

or $$x^2(a^2-h^2)-2hkxy+y^2(a^2-k^2)=0 \qquad \ldots(2)$$

Now OT_1 will be at right angles to OT_2

if sum of the coefficients of x^2 and y^2 of (2) is zero ...(Article 7·4 cor.)

i.e., if $(a^2-h^2)+(a^2-k^2)=0$

or $$h^2+k^2=2a^2$$

Hence the locus of P(h,k) is

$$x^2+y^2=2a^2$$

which is also a circle.

Exercise 14

Determine the position of :

1. the point (6, 9) *w.r.t.* the circle $x^2+y^2=100$.

2. the point (7, -11) *w.r.t.* the circle $x^2+y^2=10x$.

3. the point (2, 3) *w.r.t.* the circle

$$x^2+y^2-3x+2y+5=0.$$

4. the point ($a\cos\theta$, $a\sin\theta$) *w.r t.* the circle

$$x^2+y^2=a^2.$$

5. the point (0, 0) *w.r.t.* the circle

$$x^2+y^2+4x+y=0.$$

6. the point (1, 1) *w.r.t.* the circle

$$x^2+y^2-5x-7y+1=0.$$

7. Find the equations of the tangents drawn from the point (4, -2) to the circle

$$x^2+y^2=10$$

8. Find the equations of the tangents drawn from the point (4, 2) to the circle $x^2+y^2=4$. Find also the co-ordinates of their points of contact.

9. Find the equations of the tangents from the point (0, 1) to the circle $x^2+y^2-2x+4y=0$.

Find the lengths of the tangents drawn

10. to the circle $2x^2+2y^2=3$ from the point (-2, 3).

11. to the circle $3x^2+3y^2-7x-6y=12$ from the point (6, -7).

12. to the circle $x^2+y^2+2bx-3b^2=0$ from the point ($a+b$, $a-b$).

Find the equation of the pair of tangents drawn

13. from the point (11, 5) to the circle

$$x^2+y^2=65$$

14. from the point (4, 5) to the circle

$$2x^2+2y^2-8x+12y+21=0.$$

15. Find the equation to the chord of the circle $x^2+y^2=9$ which is bisected at the point $(-1, 2)$.

16. Find the equation to the chord of the circle $x^2+y^2=4$ whose middle point is $(-2, 3)$.

17. Prove that a chord of a circle is at right angles to the diameter through its middle point.

18. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which pass through the given point (h, k).

19. Prove that the locus of the middle points of the chords of the circle $x^2+y^2-2x-6y=10$ which pass through the origin is also a circle.

20. Find the equation of the chord of contact of tangents from the point $(3, 4)$ to the circle

$$x^2+y^2+4x+10y+4=0.$$

21. Tangents are drawn to the circle $x^2+y^2=a^2$ at the points where $x \cos \alpha+y \sin \alpha=p$ meets it. Show that their point of intersection is $\left(\dfrac{a^2 \cos \alpha}{p}, \dfrac{a^2 \sin \alpha}{p}\right)$.

22. Tangents are drawn to the circle $x^2+y^2=a^2$ from a variable point P on the line $x=b$. Prove that the locus of the middle points of the chords of contact is the circle

$$b(x^2+y^2)=a^2x.$$

Find the polar of the point :

23. $(1, 2)$ *w.r.t.* the circle $x^2+y^2=7$.

24. $(4, -1)$ *w.r.t.* the circle $2x^2+2y^2=11$.

25. $(-2, 3)$ *w.r.t.* the circle $x^2+y^2-4x-6y+5=0$.

26. $(a, -b)$ *w.r.t.* the circle

$$x^2+y^2+2ax-2by+a^2-b^2=0.$$

Find the pole of the straight line :

27. $x+2y=1$ *w.r.t.* the circle $x^2+y^2=5$.

28. $2x-y=6$ *w.r.t.* the circle $5x^2+5y^2=9$.

29. $2x+y+12=0$ *w.r.t.* the circle

$$x^2+y^2-4x+3y-1=0.$$

30. Find the condition that the chord of contact of tangents from the point (x', y') to the circle $x^2+y^2=a^2$ should subtend a right angle at the centre.

31. Prove that if the pole of the straight line *w.r.t.* the circle $x^2+y^2=a^2$ lies on the circle $x^2+y^2=9a^2$, the line is a tangent to the circle $9x^2+9y^2=a^2$.

32. The chord of contact of tangents drawn from any point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that a, b, c are in G.P.

33. Show that the lines $2x+3y=12$ and $3x+2y=2$ are conjugate *w.r.t.* the circle $x^2+y^2=2$.

34. Show that the points $(-2, -3)$ and $(-4, 1)$ are conjugate *w.r.t.* the circle $x^2+y^2=5$.

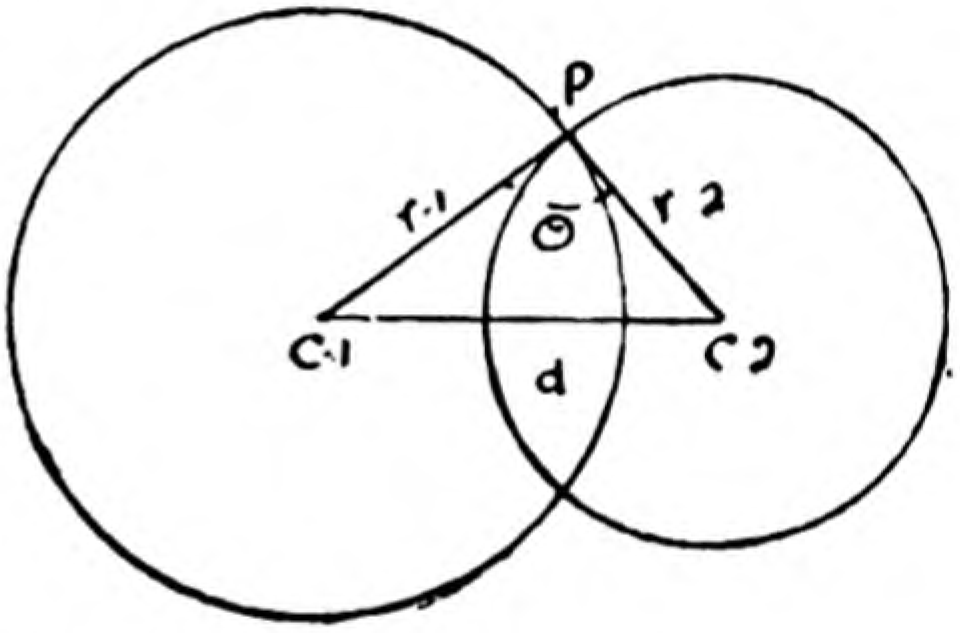
CHAPTER XI

TWO OR MORE CIRCLES

Angle of Intersection of two Circles

11.1. Def. *The angle of intersection of two circles is the angle between their tangents at a point of intersection.*

To find the angle of intersection of two circles whose radii are r_1 and r_2 and d the distance between their centres.



Let C_1, C_2 be the centres of the two circles and P the point of intersection. Join C_1P, C_2P, and C_1C_2.

Then $C_1P = r_1$, $C_2P = r_2$, and $C_1C_2 = d$.

Now let $\angle C_1PC_2 = \theta$, then from the $\triangle C_1PC_2$, by *cosine formula*, we have

$$\cos\theta = \frac{C_1P^2 + C_2P^2 - C_1C_2^2}{2C_1P.C_2P}$$

$$= \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

which gives

$$\theta = \cos^{-1} = \frac{\mathbf{r_1^2 + r_2^2 - d^2}}{\mathbf{2r_1r_2}}$$ as the required angles

Orthogonal Circles

11.2. Def. *Two circles are said to be orthogonal if their angles of intersection is a right angle.*

Such circles are said to cut each other orthogonally.

11.3. *To find the condition that the two circles*

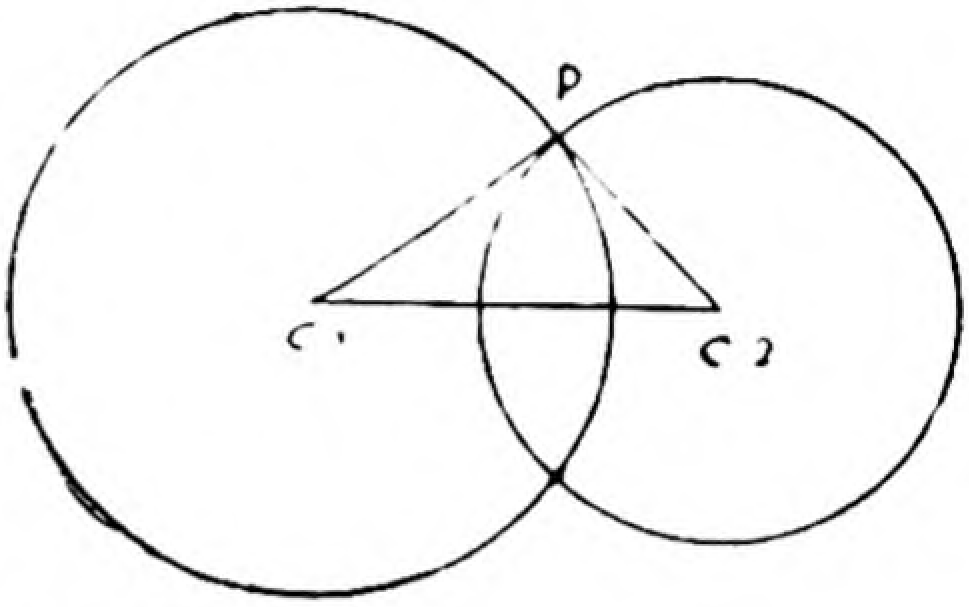
$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0$$

may cut orthogonally.

Let $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ be the centres of the two circles, and P their point of intersection, so that C_1P is the radius of the first circle and C_2P the radius of the other.



Now if the two circles cut orthogonally, $\angle C_1PC_2 = 90°$, which gives

$$C_1C_2^2 = C_1P^2 + C_2P^2$$

or $$[-g_1-(-g_2]^2+[\ f_1-(-f_2)]^2=(\sqrt{g_1^2+f_1^2-c_1})^2 + (\sqrt{g_2^2+f_2^2-c_2})^2$$

or $$(g_2-g_1)^2+(f_2-f_1)^2=g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2$$

or $$-2g_1g_2-2f_1f_2=-c_1-c_2$$

or $$\mathbf{2g_1g_2+f_1f_2=c_1+c_2}$$

which is the required condition.

Intersection of Two Circles

11.4. *To find the points of intersection of the circles*

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0$$

The equations of the circles are

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \dots(1)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \dots(2)$$

As a rule, the co-ordinates of the points of intersection can be found by solving (1) and (2) simultaneously.

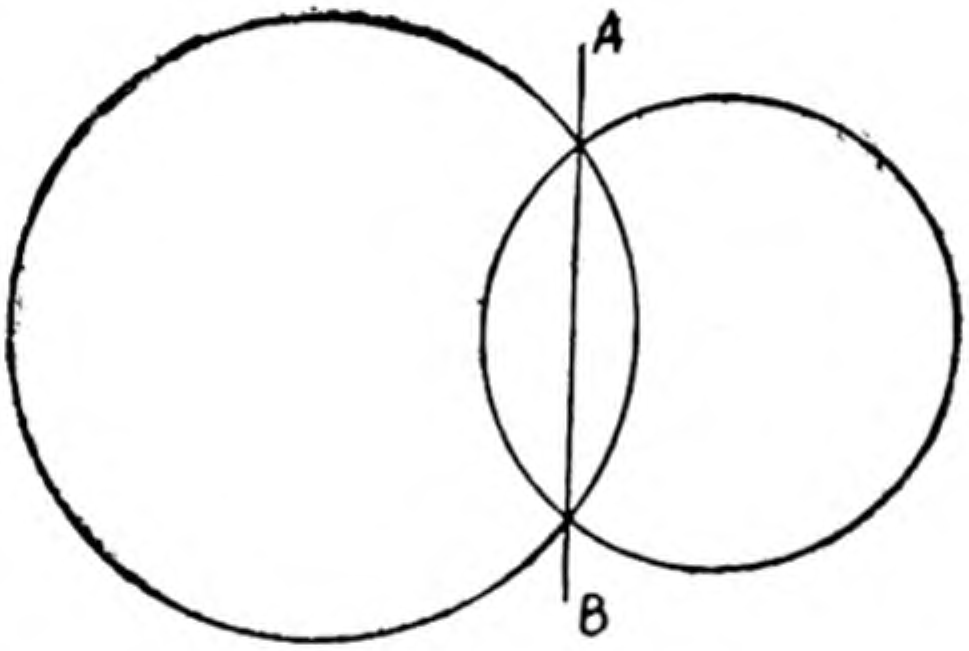
∴ Subtracting (2) from (1) we get

$$2x(g_1-g_2)+2y(f_1-f_2)+(c_1-c_2)=0 \qquad \dots(3)$$

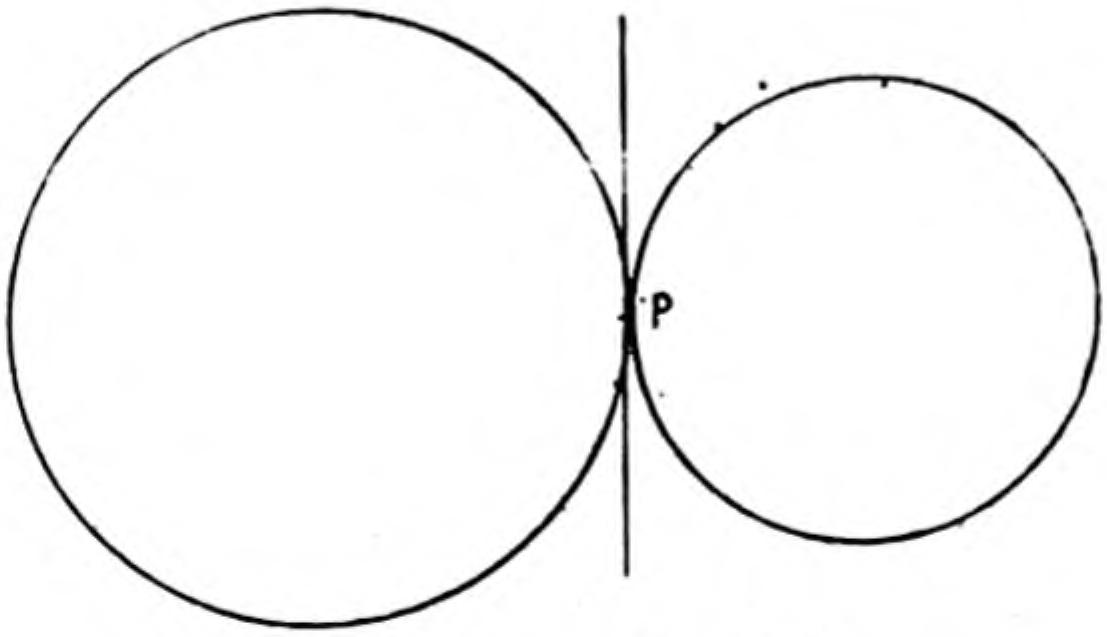
Now by solving (1) and (3), or (2) and (3) we can get the required co-ordinates. It is obvious that the two circles will intersect in **two** points. These points of intersection may be *real*, *coincident*, or *imaginary*. Two *touching* circles intersect in two *coincident* points.

11.5. Common Chord.

The co-ordinates of two intersecting circles (1) and (2) in Article 11.4 satisfy even (.) which is a st. line and is called the **common chord** of the two intersecting circles. If the two circles **touch** each other, then this common chord becomes the **common tangent** to the two circles at the point of contact.



Case of Common Chord



Case of Common tangent.

Note : *There can be no common chord or common tangent in the case of two circles intersecting in two imaginary points.*

Systems of Circles

11.6. *To find the equation of any circle through the points of intersection of the circle* $x^2+y^2+2gx+2fy+c=0$ *and the line* $Ax+By+C=0$

The equution of the circle is

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(1)$$

and that of the line is

$$Ax+By+C=0 \qquad \ldots(2)$$

Let us consider the following equation :—

$$x^2+y^2+2gx+2fy+c+k(Bx+By+C)=0 \qquad \ldots(3)$$

Now in equation (3) we notice that it is of second degree in x and y, the coefficient of x^2 is equal to the coefficient of y^2, and there is no term like xy in it. Hence it represents a circle. Moreover, the co-ordinates of points which satisfy (1) and (2) will *obviously* satisfy (3) also. Therefore, the point of intersection of the circle (1) and of the line (2) lie on (3).

Hence (3) represents the equation of any circle passing through the points of intersection of (1) and (2).

Note : The value of k can be found from the second condition satisfied by the circle.

11.7. *To find the equation of any circle through the points of intersection of the two circles*

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0$$

The equations of the two circles are :

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \ldots(1)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \ldots(2)$$

Let us consider the following equation :

$$x^2+y^2+2g_1x+2f_1y+c_1+k(x^2+y^2+2g_2x+2f_2y+c_2)=0 \qquad \ldots(3)$$

Now in equation (3) we notice that it is of second degree in x and y, the coefficient of x^2 is equal to the coefficient of y^2, $(1+k)$ in each case and it does not involve the product of xy.

Therefore, it represents a circle. Moreover, the co-ordinates of the points which satisfy (1) and (2) will *obviously* satisfy (3) also. Hence it represents a circle passing through the points of intersection of the circles (1) and (2).

Note : (1) The value of k can be found from the second condition satisfied by the circle.

(2) *In practice, it is preferable to find the equation of the common chord of the two circles, and then find the equation of any circle passing through the intersection of one of the circles and their common chord.*

Solved Examples

Ex. 1. Find the angle at which the circles

$$x^2+y^2-2x-4-4y=0$$

and

$$x^2+y^2=16$$

intersect.

Sol. Let r_1 and r_2 be the radii of these two circles and d the distance between their centres, so that

$$r_1=\sqrt{1+4+4}=3,\quad r_2=4$$

and

$$d=\sqrt{(-1-0)^2+(-2-0)^2}=\sqrt{1+4}=\sqrt{5}$$

$\therefore$ Using $\cos\theta=\dfrac{r_1{}^2+r_2{}^2-d^2}{2r_1r_2}$, we get

$$\cos\theta=\frac{9+16-5}{2.3.4}=\frac{20}{24}=\frac{5}{6}$$

which gives

$$\theta=\cos^{-1}\frac{5}{6}$$

Ex. 2. Show that the circles

$$x^2+y^2-4x-6y-12=0$$

and

$$x^2+y^2+6x+4y-12=0$$

are orthogonal to each other.

Sol. Comparing the given circles with

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $$x^2+y^2+2g_2x+2f_2y-c_2=0$$

we get

$$g_1=-2, f_1=-3$$
$$c_1=-12$$

and $$g_2=3, f_2=2, c_2=-12$$

Now $$2g_1g_2+2f_1f_2 \quad =2\,[-2\times3-3\times2]$$
$$=2(-6-6)=2(-12)=-24$$

and $$c_1+c_2=-12-12=-24 \qquad \text{...(Art. 11.3)}$$

$\therefore$ $$2g_1g_2+2f_1f_2=c_1+c_2$$

Hence the two given circles are orthogonal.

Ex. 3. Find the equation and the length of the common chord of the circles

$$x^2+y^2-5x-7y+2=0 \text{ and } x^2+y^2-7x-3y+2=0$$

Sol. The equations of the circles are

$$x^2+y^2-5x-7y+2=0 \qquad \text{...(1)}$$
$$x^2+y^2-7x-3y+2=0 \qquad \text{...(2)}$$

(*i*) Subtracting (2) from (1) we get

$$2x-4y=0$$

or $$x-2y=0$$

which gives the equation of the common chord. (Art. 11.5)

(*ii*) Next, centre of the circle (1) is $(\frac{5}{2}, \frac{7}{2})$, and the radius

$$=\sqrt{\frac{25}{4}+\frac{49}{4}-2}=\frac{\sqrt{66}}{2}$$

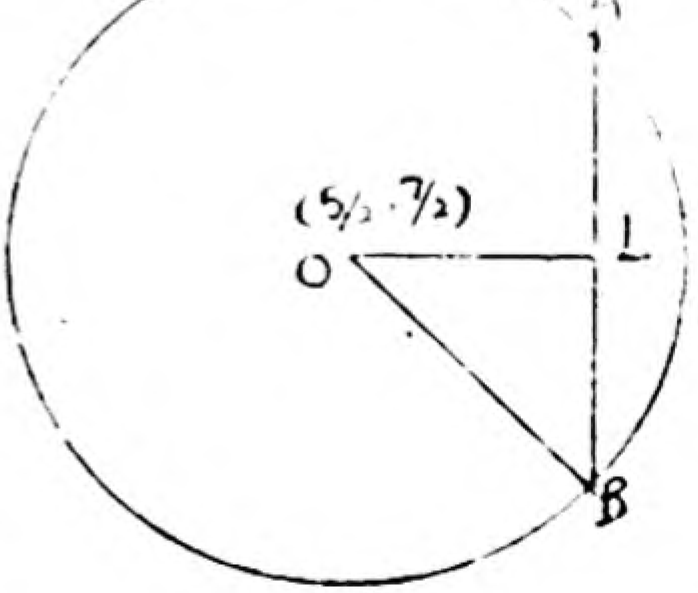
Let O be the centre of the circle (1) and AB the common chord whose equation is $x-2y=0$.

Draw OL $\perp$ on AB. Then

$$\text{OL}=\frac{\frac{5}{2}-2.\frac{7}{2}}{\sqrt{1+4}}=\frac{\frac{5}{2}-7}{\sqrt{5}}$$

$$=-\frac{9}{2\sqrt{5}}=\frac{}{2\sqrt{5}}$$

(neglecting $-$ve sign)



$$\text{OB}=\text{radius of the circle}=\frac{\sqrt{66}}{2}$$

$$\therefore \quad \text{BL}=\sqrt{\text{OB}^2-\text{OL}^2}=\sqrt{\frac{66}{4}-\frac{81}{20}}=\sqrt{\frac{249}{20}}$$

$$=\tfrac{1}{2}\sqrt{\frac{249}{5}}$$

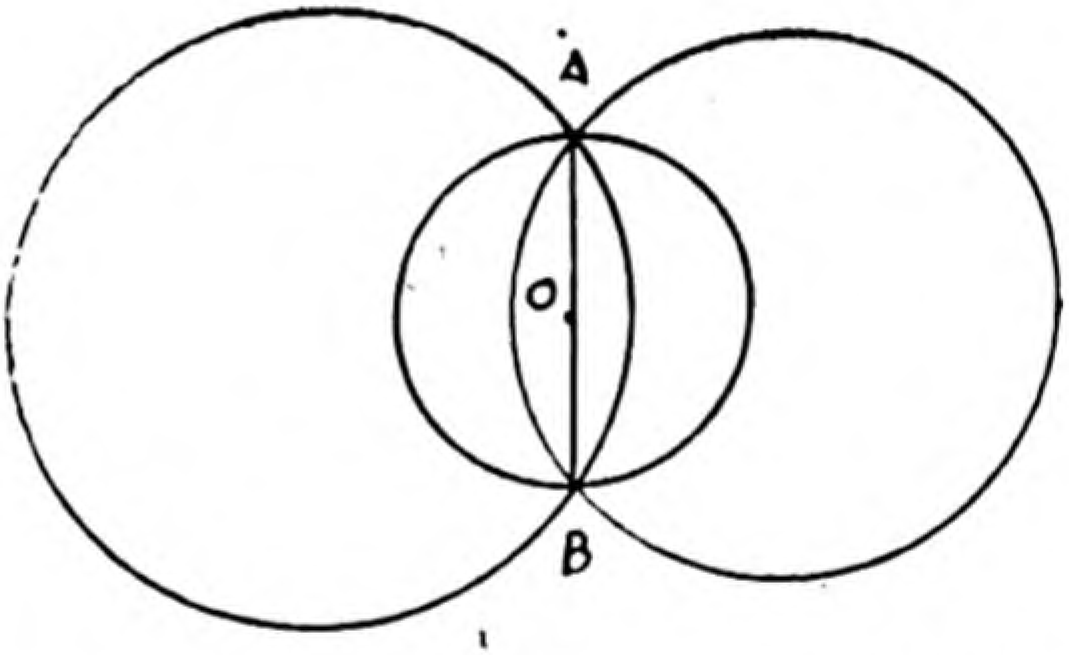
Hence $\text{AB}=2\text{BL}=\sqrt{\dfrac{249}{5}}$

Ex. 4. Find the equation of the circle which has for its diameter the common chord of the circles

$$x^2+y^2+4x+5y+1=0 \qquad \ldots(1)$$

and

$$x^2+y^2-6x+3y+2=0 \qquad \ldots(2)$$



Sol. The equation of the common chord AB is

$$10x+2y-1=0 \qquad \ldots(3)$$

The equation of the circle passing through the common points of (1) and (2) is given by

$$x^2+y^2+4x+5y+1+k(x^2+y^2-6x+3y+2)=0$$

(Article 11.7)

or $(1+k)x^2+(1+k)y^2+(4-6k)x+(5+3k)y+(1+2k)=0$

or $x^2+y^2+\dfrac{4-6k}{1+k}x+\dfrac{5+3k}{1+k}y+\dfrac{1+2k}{1+k}=0 \qquad \ldots(4)$

The centre of circle (4) is $\left[-\frac{2-3k}{1+k}, -\frac{5+3k}{2(1+k)}\right]$

It lies on (3)

$$\therefore \quad -\frac{10(2-3k)}{1+k}-\frac{2(5+3k)}{2(1+k)}-1=0$$

or $\quad -10(2-3k)-(5+3k)-(1+k)=0$

or $\quad 30k-3k-k=20+5+1$

or $\quad 26k=26$

or $\quad k=1.$

Hence from (4) the equation of the circle is

$$x^2+y^2+\frac{4-6}{1+1}x+\frac{5+3}{1+1}y+\frac{1+2}{1+1}=0$$

or $\quad 2(x^2+y^2)-2x+8y+3=0$

Exercise 15

Find the angle at which the following circles intersect :—

1. $x^2+y^2-2x-4y=4$ and $x^2+y^2=16$.
2. $(x-a)^2+(x-b)^2=h^2$ and $(x-c)^2+(x-d)^2=k^2$.

Show that the following pairs of circles cut orthogonally :—

3. $x^2+y-4x-6y-12=0$ and $x^2+y^2+6x+4y-12=0$.
4. $x^2+y^2+6x-8y-15=0$ and $x^2+y^2-2x+10y-31=0$.
5. $x^2+y^2+2x-4y+3=0$ and $x^2+y^2-6x+4y-17=0$.
6. Find the condition that the circles

$$\lambda x^2+\lambda y^2+2ax+2by+c=0 \text{ and } x^2+y^2+2a'x+2b'y+c'=0.$$

may be orthogonal. *(P.U. 1954)*

7. If $S=0$ and $S'=0$ be two circles of radii "a" and "a'" respectively, prove that the circles

$$\frac{S}{a}\pm\frac{S'}{a'}=0$$

intersect at right angles.

8. Find the equation of a circle passing through the origin, having its centre on the line $x+y=4$ and cutting the circle

$$x^2+y^2-4x+2y+4=0 \text{ orthogonally.}$$

9. Find the equation of the circle through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2+2y^2+5x-6y+4=0$.

10. Prove that the equation of the circle of which the diameter is the chord cut off by the line $x\cos\alpha+y\sin\alpha=p$ on the circle $x^2+y^2=a^2$ is

$$x^2+y^2-a^2=2p(x\cos\alpha+y\sin\alpha-p)$$

(*K.U., B.A., 1957*)

11. Find the equations of the two circles which intersect the circles

$$x^2+y^2-6y+1=0$$

and

$$x^2+y^2-4y+1=0$$

orthogonally and touch the line

$$3x+4y+5=0.$$

12. Find the equation to the circle which cuts orthogonally each of the three circles

$$x^2+y^2+2x+17y+4=0$$
$$x^2+y^2+7x+6y+11=0$$
$$x^2+y^2-x+22y+3=0.$$

Find the lengths of the common chords of the circles

13. $x^2+y^2+ax+by+c=0$ and $x^2+y^2+bx+ay+c=0$.

14. $x^2+y^2-2y-9=0$ and $2x^2+2y^2+3x+y-6=0$.

15. Find the equation of the circle whose diameter is the common chord of the circles

$$x^2+y^2+2x+3y+1=0 \text{ and}$$
$$x^2+y^2+4x+3y+2=0.$$

16. Find the length of the common chord of the circles $(x-a)^2+y=a^2$ and $x^2+(y-b)^2=b^2$ and prove that the equation of the circle whose diameter is this common chord is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay)$$

17. Find the equation of the circle

(*i*) passing through the intersection of $2x+y-10=0$ and $x^2+y^2=25$ and the origin.

(*ii*) passing through the intersection of the circles

$$x^2+y^2+2x+3y-7=0 \text{ and } x^2+y^2+3x-2y-1=0$$

and through the point (1, 2).

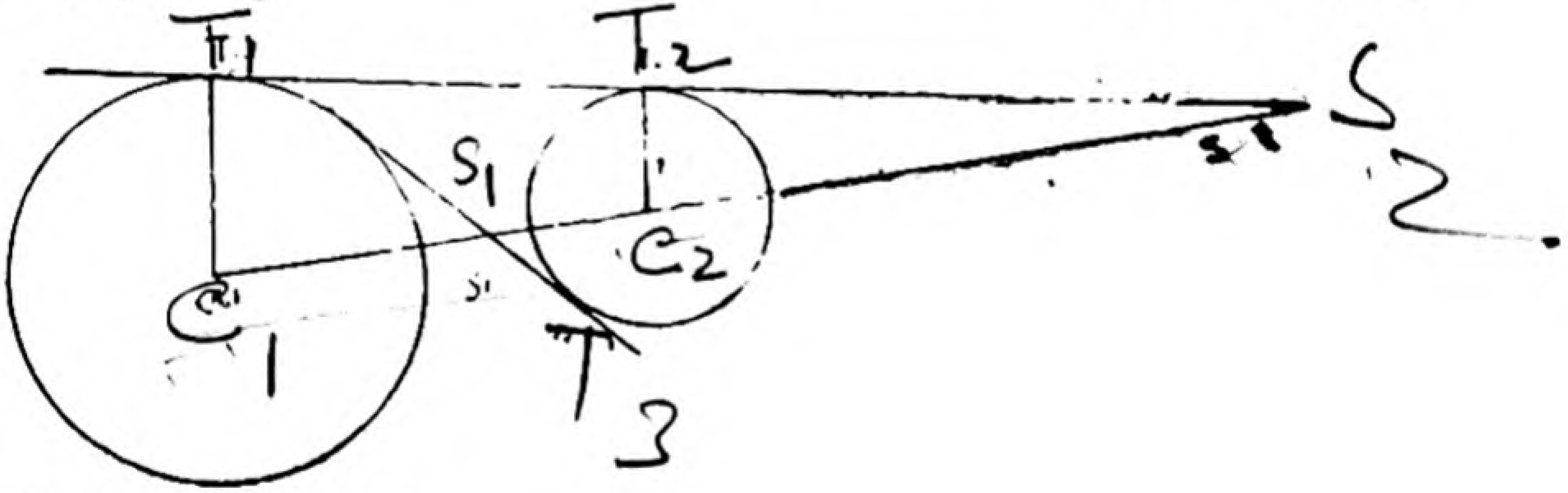
(*iii*) passing through the intersection of the circles $x^2+y^2=2ax$ and $x^2+y^2=2by$ and having its centre on the line

$$\frac{x}{a}-\frac{y}{b}=2.$$

11 8. Common tangents to two circles

Let C_1, C_2 be the centres of the two circles, and r_1, r_2 be their radii.

Draw the *direct* common tangent T_1T_2 and let it meet C_1C_2 in S_2. Similarly, let the *transverse* common tangent T_1T_3 meet C_1C_2 in S_1.



Join C_1T_1 and C_2T_2.

Now from the *similar* △s $C_1T_1S_2$ and $C_2T_2S_2$, we have

$$\frac{C_1S_2}{C_2S_2}=\frac{C_1T_1}{C_2T_2}=\frac{r_1}{r_2}$$

∴ S_2 divides C_1C_2 *externally* in the ratio of the *radii*.

Similarly, it can be shown that S_1 divides C_1C_2 *internally* in the ratio of the *radii*.

The two points S_1 and S_2 are known as the **internal and external centres of similitude.**

The pair of tangents from S_1 are the *transverse common tangents* whereas those from S_2 are the *direct common tangents.*

Hence the following method :—

(*i*) *Find the co-ordinates of* S_1 *and* S_2 *which divide the line of centres internally and externally in the ratio of the radii.*

(*ii*) *Write down the equations of the tangents through* S_1 *and* S_2 *to either of the two circles.*

Illustration :—

Find the common tangents to the circles

$$x^2+y^2=1 \qquad \ldots(1)$$

and $$(x-1)^2+(y-3)^2=4 \qquad \ldots(2)$$

Sol. The centres of the circles are (0, 0) and (1, 3).

Their radii are 1 and 2.

Now S_1, the point which divides the line joining (0, 0) and (1, 3) internally in the ratio 1 : 2 is given by

$$S_1=\left(\frac{1\times1+2\times0}{1+2},\ \frac{1\times3+2\times0}{1+2}\right)=(\tfrac{1}{3},\ 1)$$

Similarly, S_2, the point which divides the same line externally in the ratio 1 : 2 is given by

$$S_2=\left(\frac{1\times1-2\times0}{1-2},\ \frac{1\times3-2\times0}{1-2}\right)=(-1,\ -3)$$

Now, any tangent to circle (1) is

$$y=mx+\sqrt{1+m^2} \qquad \ldots(3)$$

It passes through $(\frac{1}{3}, 1)$

$$\therefore \qquad 1=\tfrac{1}{3}m+\sqrt{1+m^2}$$

or $$(1-\tfrac{1}{3}m)^2=1+m^2$$

or $$1+\tfrac{1}{9}m^2-\tfrac{2}{3}m=1+m^2$$

or $$4m^2+3m=0 \qquad \text{which gives}$$

$$m=0,\quad m=-\tfrac{3}{4}.$$

$\therefore$ (3) gives

$$y=1 \quad \text{and} \quad y=-\tfrac{3}{4}x+\tfrac{5}{4}$$

or $$y-1=0 \quad \text{and} \quad 3x+4y=5$$

These give the equations of the indirect (or transverse) common tangents.

Again, (3) passes through $(-1, -3)$

$\therefore$ $$-3=-m+\sqrt{1+m^2}$$

or $$(m-3)^2=1+m^2$$

or $$m^2-6m+9=1+m^2$$

or $$0.m^2+6m-8=0$$ (Please note this step)

This gives $m=\frac{4}{3}$ and ∞ ($\because$ Coeff. of m^2 is zero)

If $m=\frac{4}{3}$ then (3) becomes

$$y=\tfrac{4}{3}x+\sqrt{1+\tfrac{16}{9}}$$

or $$y=\tfrac{4}{3}x+\tfrac{5}{3}$$

or $4x-3y+5=0$

Again the tangent with m as ∞ is parallel to the y-axis. As it passes through $(-1, -3)$ its equation is $x=-1$

Hence $$4x-3y+5=0$$

and $$x+1=0$$

are the two direct common tangents.

Radical Axis

11.9. Definition : *The radical axis of two cricles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.*

Let the equations of the circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \text{...(1)}$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \text{...(2)}$$

Let $P(x', y')$ be *any* point such that the tangents from it to these two circles are *equal.* Let PT and PT$'$ be these tangents.

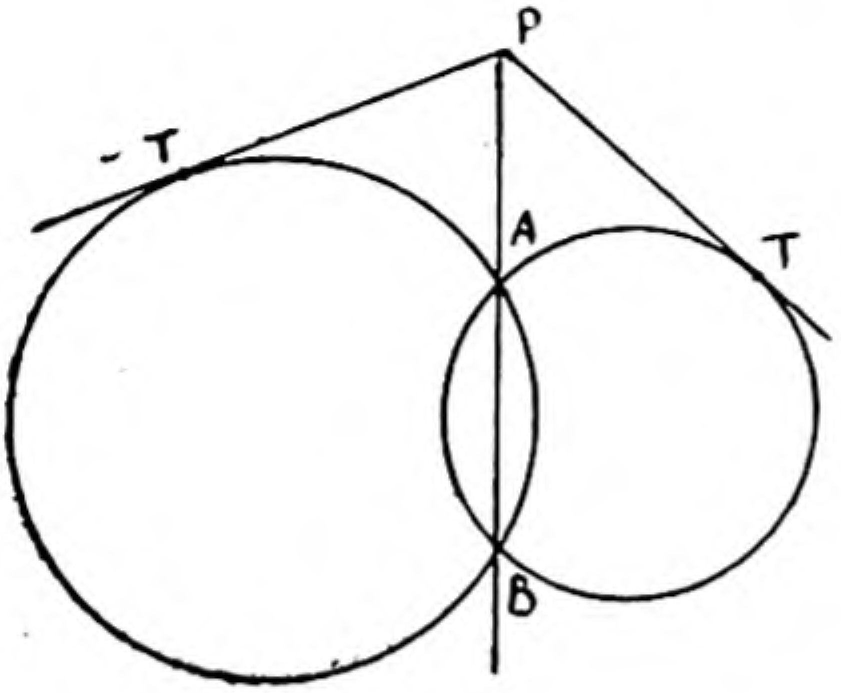


Fig. 1

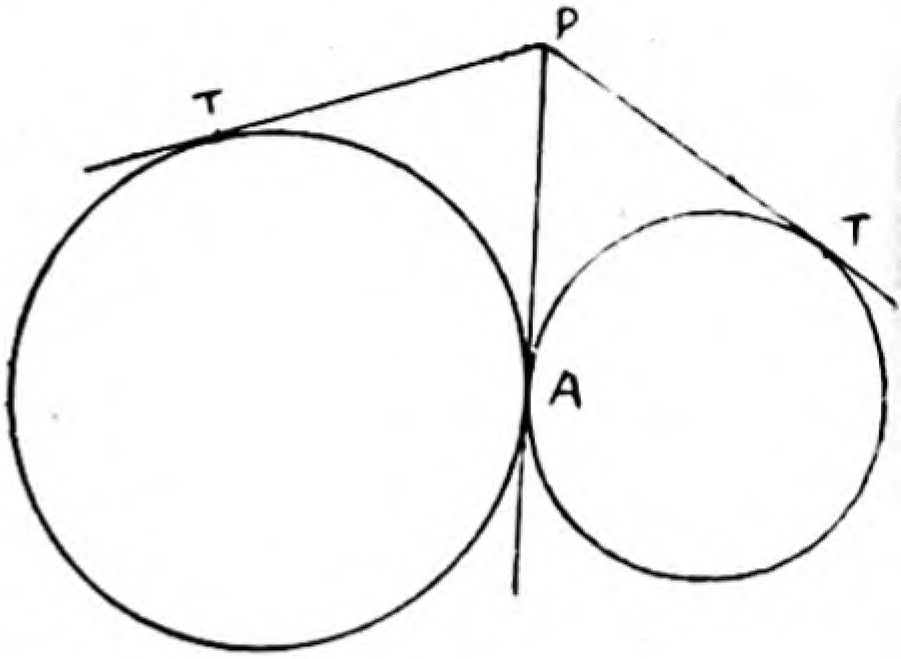


Fig 2.

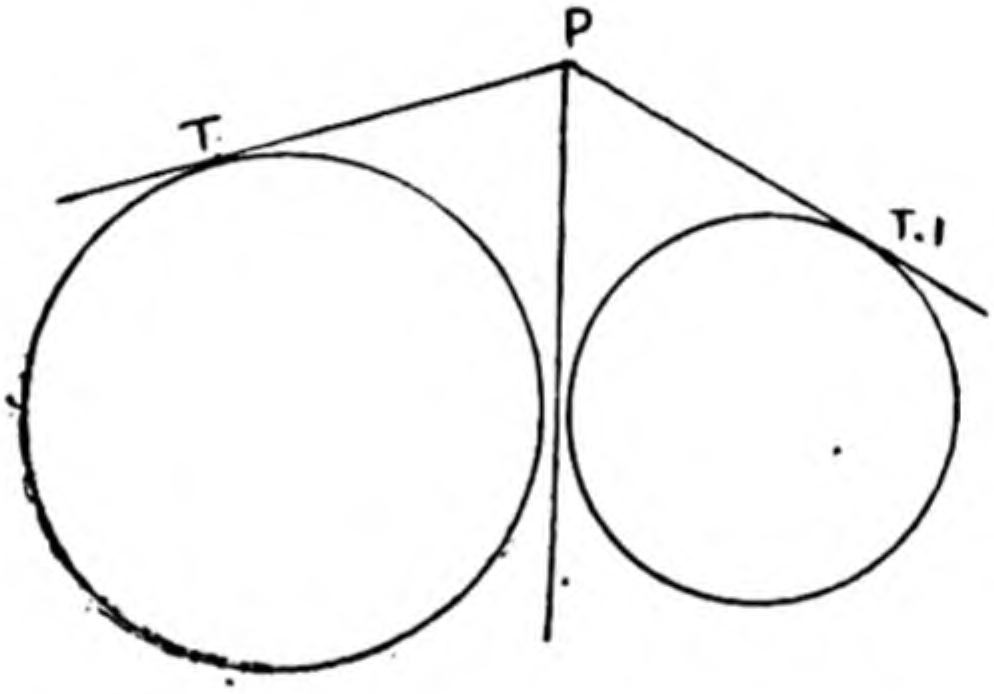


Fig 3.

Then $\quad PT=\sqrt{x'^2+y'^2+2g_1x'+2f_1y'+c_1}$

and $\quad PT'=\sqrt{x'^2+y'^2+2g_2x'+2f_2y'+c_2}$

(Article 10.7)

Now $\quad PT=PT'$

$\therefore$

$$x'^2+y'^2+2g_1x'+2f_1y'+c_1$$
$$=x'^2+y'^2+2g_2x'+2f_2y'+c_2$$

or $\quad 2x'(g_1-g_2)+2y'(f_1-f_2)+(c_1-c_2)=0$

$\therefore$ the locus of $P(x', y')$ is

$$\mathbf{2x(g_1-g_2)+2y(f_1-f_2)+(c_1-c_2)=0} \qquad \ldots(3)$$

which gives the equation of the Radical Axis.

Note : (1) *The radical axis of two circles is a straight line because* (3) *is a first degree equation in* x, y.

(2) *If the equations of two circles are denoted by* $S_1=0$ *and* $S_2=0$ *then the equation of their radical axis is* $S_1-S_2=0$ *provided that the equations of the two circles are written such that the coefficients of* x^2 *and* y^2 *in each are* $=$ *unity.*

(3) *If the two circles* $S_1=0$ *and* $S_2=0$ *intersect, then their radical axis is the same as their common chord. If the circles touch, then the radical axis is their common tangent at their point of contact. This is clear from figures No.* (*i*) *and* (*ii*) *respectively.*

11.10. Theorems on Radical Axis

Theorem I. *The radical axis of two circles is perpendicular to the line joining their centres.*

Let the two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$x^2+y^2+2g_2x+2f_2y+c_2=0$$

The centres of these circles are

$$(-g_1, -f_1) \text{ and } (-g_2, -f_2)$$

$\therefore$ the slope of the line joining these centres

$$=\frac{-f_2-(-f_1)}{-g_2-(-g_1)}=\frac{f_1-f_2}{g_1-g_2} \qquad \ldots(1)$$

$$\left(\because\ m=\frac{y_2-y_1}{x_2-x_1}\right)$$

Now the equation of the radical axis of the two given circles is

$$2x(g_1-g_2)+2y(f_1-f_2)+(c_1-c_2)=0$$

(Article 11.9)

$\therefore$ the slope of the radical axis is

$$=-\frac{2(g_1-g_2)}{2(f_1-f_2)}=-\frac{g_1-g_2}{f_1-f_2} \qquad \ldots(2)$$

$$\left(\because\ m=-\frac{b}{a}\right)$$

The product of slopes (1) and (2) is

$$\frac{f_1-f_2}{g_1-g_2}\times\left(-\frac{g_1-g_2}{f_1-f_2}\right)=-1$$

This shows that the radical axis is perpendicular to the line joining the centres.

Theorem II. *The radical axes of these circles, taken in pairs, meet in a point.*

Let the equations of the circles be

$$S_1=0,\ S_2=0,\ S_3=0$$

the coefficients of x^2 and y^2 being unity in each equation.

The radical axes of these circles, taken in pairs, are

$$S_1-S_2=0 \qquad \ldots(1)$$

$$S_2-S_3=0 \qquad \ldots(2)$$

$$S_3-S_1=0 \qquad \ldots(3)$$

Adding the L.H.S.'s of (1), (2) and (3), we find that their sum vanishes. *Hence the lines represented by (1), (2), (3) are concurrent.* (Article 5.5)

Note : *The common point of the radical axes of three circles, taken in pairs, is called the* **Radical Centre**.

Theorem III. *The difference of the squares of the tangents to two circles from any point in their plane varies as the distance of the point from their radical axis.*

Let us take the x-axis as the line of the centres of the circles, and the y-axis as the radical axis. We can do so because the radical axis is $\perp$ to the line joining the centres and the y-axis is also $\perp$ to the x-axis.

$\therefore$ *the equations of the two circles can be put as

$$x^2+y^2+2g_1x+c=0$$

$$x^2+y^2+2g_2x+c=0$$

(Keep the same constant in both equations)

This is the simplest form of the equation of a circle and can be explained better at a later stage.

Now the radical axis is $2x(g_1-g_2)=0$ or $x=0$, and the centres are $(-g_1, 0)$, $(-g_2, 0)$ which evidently fulfils the conditions for the special choice of the axes.

Let (x_1, y_1) be any point in the plane of the circles. Therefore, the difference of the squares of the tangents drawn from (x_1, y_1) to the circles is equal to

$$(x_1^2+y_1^2+2g_1x_1+c)-(x_1^2+y_1^2+2g_2x_1+c)$$
$$=2x_1(g_1-g_2)$$

Let us denote this difference of the L.H.S. by "d"

$$\therefore \quad d=2x_1(g_1-g_2)$$

$$\text{or} \quad \frac{d}{x_1}=2(g_1-g_2) \quad \text{(constant)}$$

Hence d varies as x_1, the distance of (x_1, y_1) from the axis of y which we have taken to be the radical axis

Theorem IV. *If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third circle.*

Let the equation of the two circles be

$$x^2+y^2+2g_1x+c=0 \qquad \ldots(1)$$

and
$$x^2+y^2+2g_2x+c=0 \qquad \ldots(2)$$

Let the circle cutting (1) and (2) orthogonally be

$$x^2+y^2+2Gx+2Fy+C=0 \qquad \ldots(3)$$

$$\therefore \quad \left.\begin{aligned} 2Gg &=c+C \\ \text{and} \quad 2Gg_2 &=c+C \end{aligned}\right\} \qquad (\because 2g_1g_2+2f_1f_2=c_1+c_2)$$

These two equations give

$$G=0, \text{ and } C=-c$$

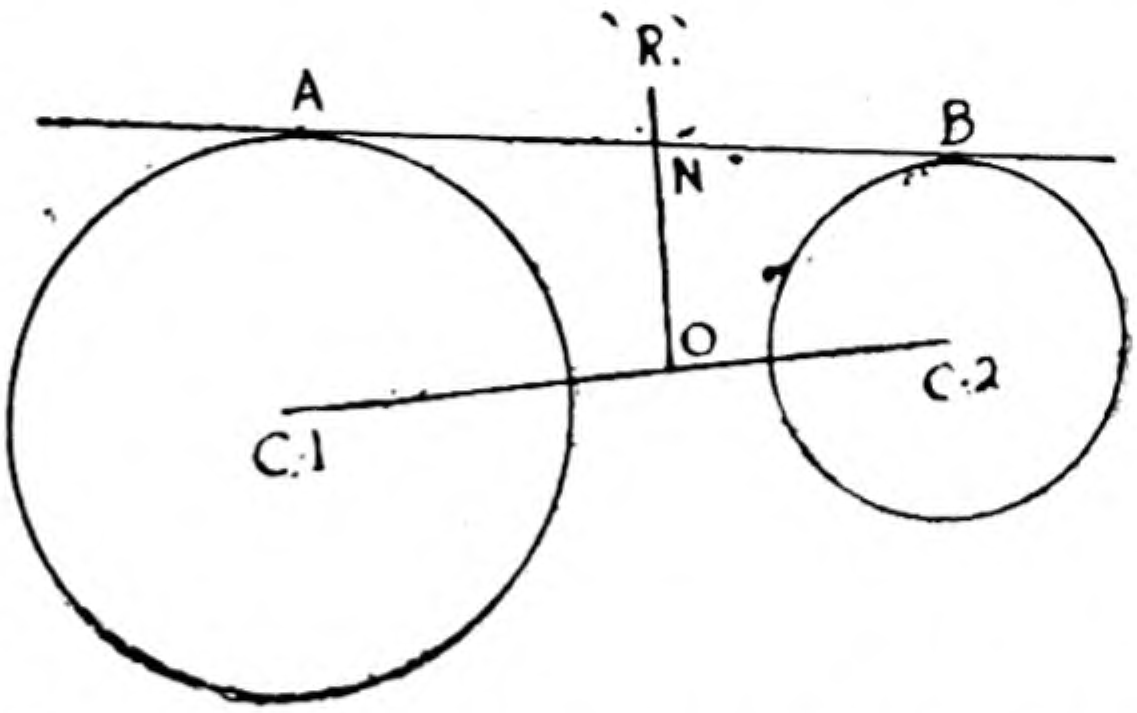
$\therefore$ the equation of the third circle becomes

$$x^2+y^2+2Fy-c=0 \qquad \ldots(4)$$

The centre of (4) is $(0, -F)$ which evidently lies on $x = 0$, the radical axis of (1) and (2).

Theorem V. *The radical axis of two circles bisects their common tangents.*

Let OR be the *radical axis* of the two circles and AB the *common tangent* cutting the radical axis in M. Then MA and MB are the tangents to the two circles from a point M on the radical axis. *Hence they are equal.*

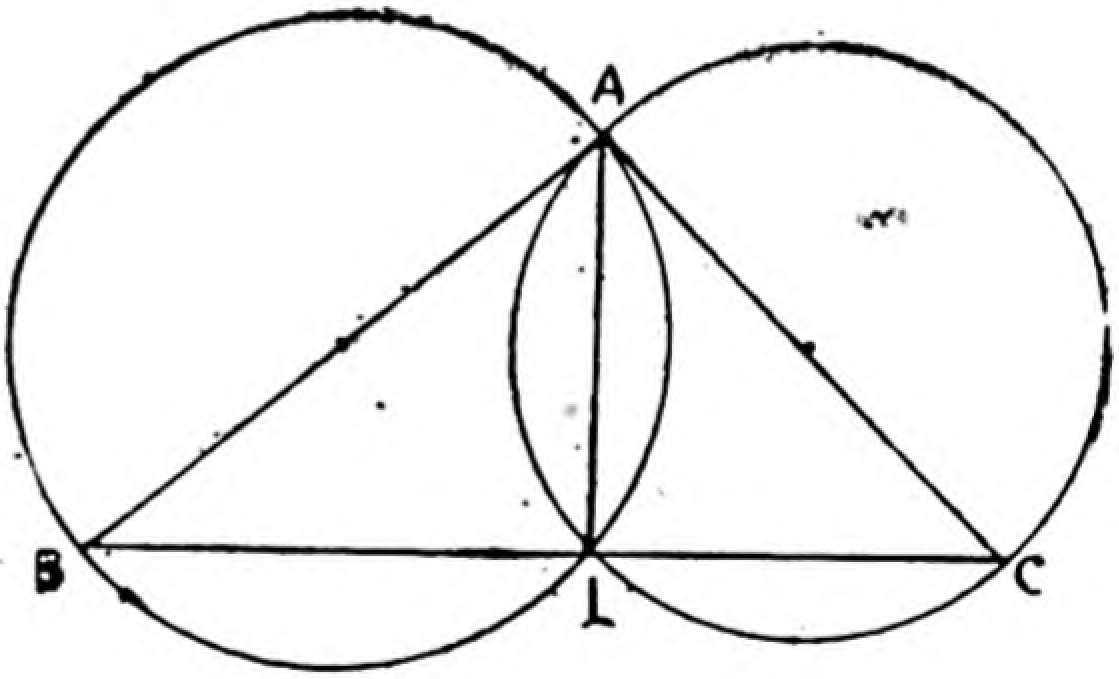


Theorem VI. *The Radical Centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.*

Let ABC be the triangle and AL the perpendicular to BC.

Then circles described on AB and AC as diameters pass through L.

Therefore, AL is that *radical axis.* Similarly, the *radical axis* of the circles on AB and BC as diameters will be the *altitude* BM, and for the third pair is the *altitude* CN. *The three radical axes, therefore, meet in the orthocentre.*



11.11. Touching Circles

Definition. *Two circles are said to touch each other if their radical axis is their common tangent at the point of contact.*

Alternatively :

Two circles are said to touch each other externally if the distance between their centres is equal to the sum of their radii, *and these are said to touch* **internally** *if the distance between their centres is equal to the difference of their radii.*

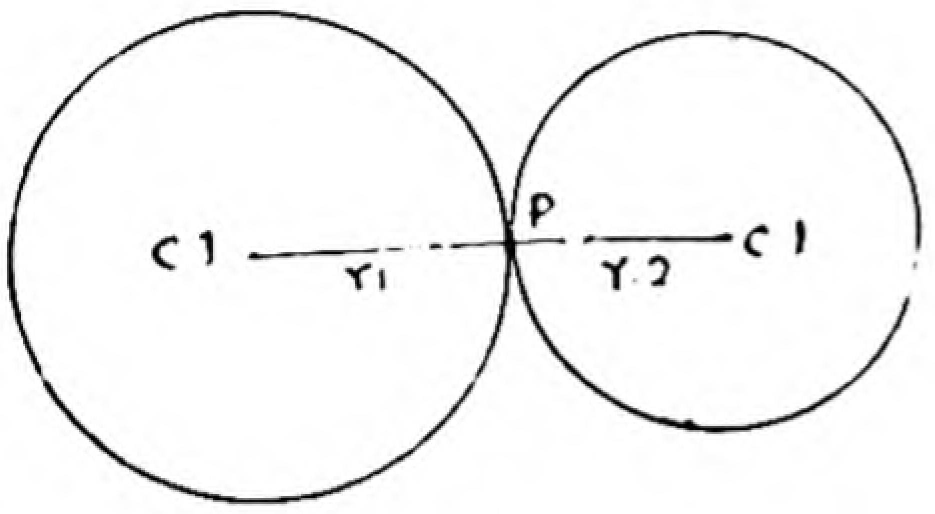


Fig 1

Case of external touch

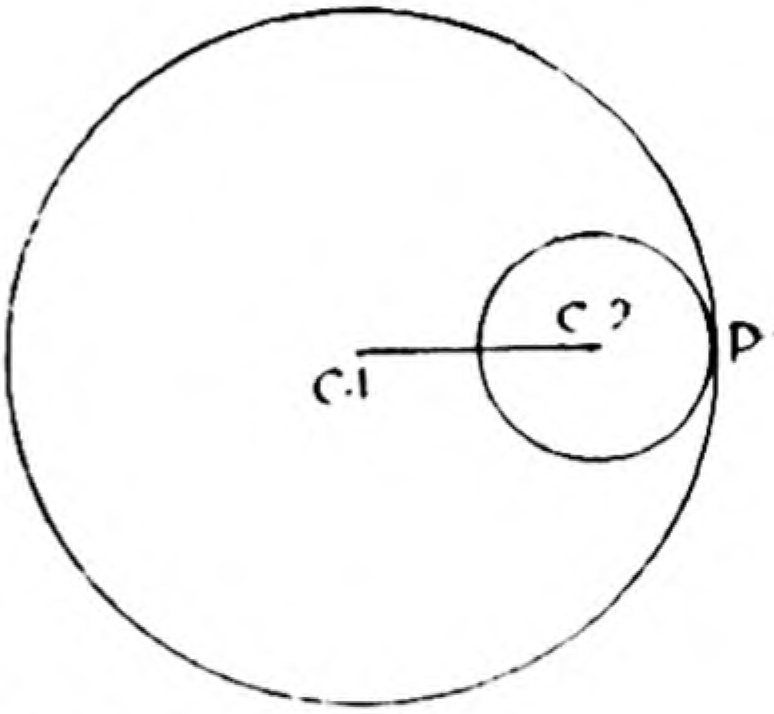


Fig 2

Case of internal touch

Co-axal Circles

11.12. Def :—*A system of circles is said to be co-axal if every pair of them has the same radical axis.*

(*a*) If $\mathbf{S=0}$ and $\mathbf{S'=0}$ be two circles then $\mathbf{S+\lambda S'=0}$ represents a **co-axal system** for *different* values of λ.

If λ_1, λ_2 be two different values of λ, then $S+\lambda_1 S'=0$ and $S+\lambda_2 S'=0$ are two circles of the system. The *radical axis* of these two circles is

$$\frac{S+\lambda_1 S'}{1+\lambda_1}-\frac{S+\lambda_2 S'}{1+\lambda_2}=0$$

which simplifies into $(\lambda_1-\lambda_2)(S-S')=0$

or $\mathbf{S-S'=0}$

which is a fixed straight line.

(*b*) If $\mathbf{S=0}$ is a circle and $\mathbf{U=0}$ is a straight line, then $\mathbf{S+\lambda U=0}$ also represents a **co-axal system** for different values of λ, $U=0$ being their radical axis.

Let λ_1, λ_2 be two different values of λ; then $S+\lambda_1 U=0$ and $S+\lambda_2 U=0$ are two circles of the system, and the radical axis of these two circles is

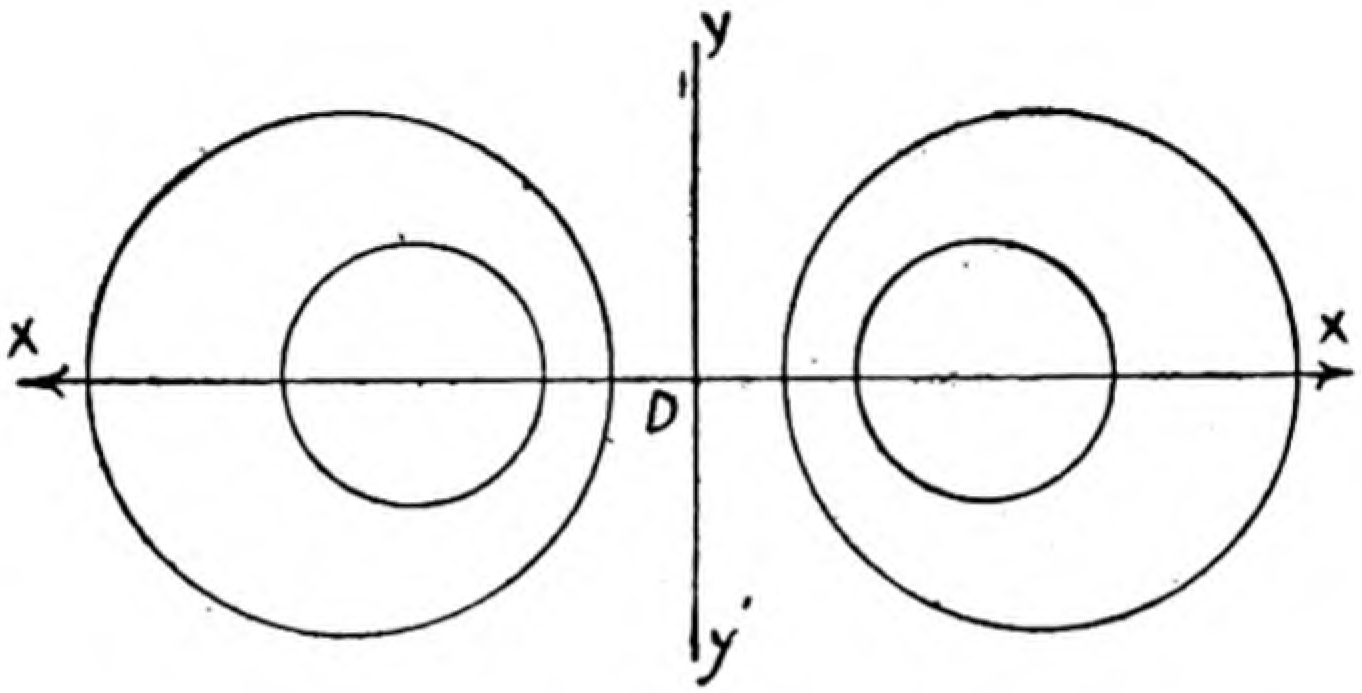
$$(S+\lambda_1 U)-(S+\lambda_2 U)=0$$

or

$$(\lambda_1-\lambda_2)U=0$$

$$\mathbf{U=0}$$

11.13. Equation to a co-axal system in the simplest form.



Let us take the line of centres as the x-axis and the common radical axis as the y-axis. We can do so because the radical axis of the two circles is always perpendicular to the line joining the centres of the circles.

Let the equations of *any* two circles of the system be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \ldots(1)$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \ldots(2)$$

Now the centres of (1) and (2) lie on x-axis.

Therefore $f_1=f_2=0$

Hence (1) and (2) become

$$x^2+y^2+2g_1x+c_1=0 \qquad \ldots(3)$$

and

$$x^2+y^2+2g_2x+c_2=0 \qquad \ldots(4)$$

The radical axis of (3) and (4) is

$$2x(g_1-g_2)+(c_1-c_2)=0 \qquad \ldots(5)$$

Now (5) is the same as $x=0$, *i.e.*, as the y-axis

$\therefore$ $c_1-c_2=0$

or $c_1=c_2=c$ (say)

$\therefore$ (3) and (4) reduce to

$$x^2+y^2+2g_1x+c=0$$

and $$x^2+y^2+2g_2x+c=0$$

where c is a constant.

From this it follows that circles of the *co axal system* are given by the equation

$$\mathbf{x^2+y^2+2gx+c=0}$$

where g is a *variable* parameter and c is a constant.

11.14. Limiting Points of a coaxal system

In Article 11.13, we have seen that the equation to a *co-axal* system of circles can be put as

$$x^2+y^2+2gx+c=0 \qquad \ldots(1)$$

where g varies.

Now (1) can be further written as

$$(x+g)^2+y^2=[\sqrt{g^2-c}]^2$$

The centre of this circle, therefore, is $(-g,\ 0)$ and the radius is $\sqrt{g^2-c}$.

This radius *vanishes*, if $g^2-c=0$ *i.e.*, if $g=\pm\sqrt{c}$. The circle then becomes a point-circle.

Hence at the particular points $(\pm\sqrt{\mathbf{c}}, \mathbf{0})$ we have **point-circles** which belong to the system. **These point-circles are called the Limiting Points of the system.**

The points $(\sqrt{\mathbf{c}}, \mathbf{0})$ **and** $(-\sqrt{\mathbf{c}}, \mathbf{0})$ which are the centres of the *point circles* belonging to the *co axal* system (1) are called the **limiting points** of this system.

Now if c is positive, the *limiting points* are *real*, but the points of intersection are *imaginary*, whereas if c is *negative*, the *limiting points* are *imaginary*, but the points of intersection are *real. This is shown below :—*

(*i*) Let any two circles of the system be

$$x^2+y^2+2g_1x+c=0 \qquad \ldots(1)$$

and

$$x^2+y^2+2g_2x+c=0 \qquad \ldots(2)$$

where c is +ve.

Subtracting (2) from (1), we get

$$x=0$$

Substituting this value of x in (1) or (2) we get

$$y^2+c=0$$

or

$$\mathbf{y=\pm\sqrt{-c}}$$

Hence the points of intersection of (1) and (2) are $\mathbf{(0, \pm\sqrt{-c})}$ and their **limiting points** are $\mathbf{(\pm\sqrt{c}, 0)}$

(*ii*) Let the circles of the system be

$$x^2+y^2+2g_1x-c=0 \qquad \ldots(1)$$

$$x^2+y^2+2g_2x-c=0 \qquad \ldots(2)$$

where c is −ve.

Subtracting (2) from (1), we get

$$x=0$$

Substituting this value of x in (1) or (2), we get

$$y^2 \quad c \qquad \text{or} \qquad \mathbf{y=\pm\sqrt{c}}$$

Hence the points of intersection of (1) and (2) are $\mathbf{(0, \pm\sqrt{c})}$, and from $\mathbf{\sqrt{g_1^2+c}=0}$ or from $\mathbf{\sqrt{g_2^2+c}=0}$, we get $\mathbf{(\pm\sqrt{-c}, 0)}$ as their **limiting points.**

11.15. Theorems on Limiting Points

Theorem I. *The limiting points of a co-axal system are conjugate w.r.t. every circle of the system.*

The equation of any circle of the system is

$$x^2+y^2+2gx+c=0$$

The *limiting points* of L $(\sqrt{c}, 0)$ and L$'(-\sqrt{c}, 0)$
(Article 11.14)

Now the polar of L($\sqrt{c}$, 0) *w.r.t.*

$$x^2+y^2+2gx+c=0 \text{ is}$$

$$\sqrt{c}x+0\,y+g(x+\sqrt{c})+c=0 \qquad \text{(Art. 10.10)}$$

or $$(\sqrt{c}x+c)+g(x+\sqrt{c})=0$$

or $$(x+\sqrt{c})(\sqrt{c}+g)=0$$

or $$x+\sqrt{c}=0 \qquad (\because g+\sqrt{c}\neq 0)$$

which *evidently* passes through L′($-\sqrt{c}$, 0)

Hence the theorem.

Theorem II. *Every circle through the limiting points cuts every member of a co-axal system orthogonally.*

Let the co-axal system be represented by

$$x^2+y^2+2gx+c=0 \qquad ...(1)$$

Let any circle passing through the limiting points ($\pm\sqrt{c}$, 0) be

$$x^2+y^2+2Ax+2By+k=0 \qquad ...(2)$$

Now that ($\pm\sqrt{c}$, 0) lie on (2), we have

$$c+2A\sqrt{c}+k=0$$

and $$c-2A\sqrt{c}+k=0$$

These give $$A=0,\ k=-c$$

$\therefore$ (2) becomes

$$x^2+y^2+2By-c=0 \qquad ...(3)$$

(Keeping B as it is)

Now the condition for (1) and (3) to be orthogonal is

$$2g(0)+2.(0).B=c-c$$

which is clearly satisfied. Hence the theorem.

Note (1) : *Circles* (1) *and* (2) *are said to be conjugate,*

Note (2) : Equation (3) can better be written as

$$\mathbf{x^2+y^2+2fy-c=0.}$$

Theorem III. *The radical axis of one system is the line of centres of the other system.*

Let the equation of the first system be

$$x^2+y^2+2gx+c=0 \qquad \ldots(1)$$

where g is a *variable*.

The radical axis of this system is

$$x=0$$

This is *clearly* the line of centre of the system

$$x^2+y^2+2fy \quad c=0 \qquad \ldots(2)$$

Again, the radical axis of system (2) is

$$y=0$$

which is *clearly* the line of centres of system (1).

This completes the proof.

Theorem IV. *The points of intersection of one system are the limiting points of the other system.*

The points of intersection of the first system, *i e.*, of $x^2+y^2+2gx+c=0$ are $(\mathbf{0} \pm \sqrt{-\mathbf{c}}\ )$ and the limiting points are $(\pm\sqrt{\mathbf{c}}, 0)$ (Article 11.14)

Now the other system is

$$x^2+y^2+2fy-c=0.$$

The *limiting* points of the system are $(\mathbf{0} \pm \sqrt{-\mathbf{c}}\ )$ and to find the points of intersection of the system let us take the following two circles belonging to the system :—

$$x^2+y^2+2f_1y-c=0 \qquad \ldots(1)$$

and

$$x^2+y^2+2f_2y-c=0 \qquad \ldots(2)$$

Subtracting (2) from (1), we get

$$y=0$$

$\therefore$ from (1), $\qquad x^2-c=0$

or $\qquad \mathbf{x}=\pm\sqrt{\mathbf{c}}$

Hence the points of intersection of this system are.

$$(\pm\sqrt{\mathbf{c}}, \mathbf{0})$$

This proves the theorem.

Solved Examples

Ex. 1. Find the radical axis of the circles

$$x^2+y^2+x-y-2=0$$

and

$$3x^2+3y^2+4x-12=0$$

Sol. The equations of the circles are

$$x^2+y^2+x-y-2=0 \quad \text{...(1)}$$

and

$$x^2+y^2+\tfrac{4}{3}x-4=0 \quad \text{...(2)}$$

(Please note this step)

Subtracting (2) from (1), we get

$$-\frac{x}{3}-y+2=0$$

or

$$x+3y=6$$

which is the radical axis.

Ex. 2. Show that the circles

$$x^2+y^2+2x+2y+1=0$$

and

$$x^2+y^2-4x-6y-3=0$$

touch each other.

Sol. (*i*) (*Method of radical axis*)

The radical axis of the two circles is

$$6x+8y+4=0$$

or

$$3x+4y+2=0$$

Now if the two circles touch, then (1) must be tangent to both the circles. If (1) touches the first circle, then the length of the perpendicular drawn from the centre of that circle, *i.e.*, from $(-1, -1)$ on (1) must be equal to its radius which is

$$\sqrt{1^2+1^2-1}=1$$

Now the ⊥ distance $=\dfrac{3\times-1+4\times-1+2}{\sqrt{9+16}}$

$$=-\frac{5}{5}=1 \qquad \textit{(in magnitude)}$$

$\therefore$ $3x+4y+2=0$ is a tangent to the first circle. *Similarly* it can be shown that (1) is the tangent to the second circle.

Hence the two circles touch :

(*ii*) (*Method of centres and radii*)

The centre of the first circle is $(-1, -1)$ and radius is 1.

The centre of the second circle is (2, 3) and the radius is $\sqrt{4+9+3}=4$.

Now if the two circles touch, then the distance between their centres must be equal to the sum or the difference of their radii, *i.e.*, if they touch *externally* the distance must be equal to the *sum* of the radii, and if they touch *internally*, it must be equal to their *difference.*

The distance between (2, 3) and $(-1, -1)$

$$=\sqrt{(2+1)^2+(3+1)^2}=\sqrt{9+16}=5$$

Obviously, this is equal to the sum of their radii.

Hence the two circles touch externally.

Ex. 3. Find the limiting points of the co-axal system determined by

$$x^2+y^2-10x-8y+5=0$$

and $$x^2+y^2+10x+2y=0$$

Sol. The radical axis of the sys'em is

$$20x+10y-5=0$$

or $$4x+2y-1=0$$

$\therefore$ the co-axal system is given by

$$x^2+y^2+10x+2y+k(4x+2y-1)=0$$

[Article 11.12 (*b*)]

or $$x^2+y^2+2(5+2k)x+2(1+k)y-k=0 \qquad ...(1)$$

The centre is $[-(5+2k), -(1+k)]$

and the radius $=\sqrt{(5+2k)^2+(1+k)^2+k}$

$$=\sqrt{5k^2+23k+26}$$

If the radius is zero, then

$$5k^2+23k+26=0$$

This gives $k=-2,\ -\frac{13}{5}.$

Substituting these values of k in (1), we get the required limiting points as

$$(-1,\ 1) \text{ and } (\tfrac{1}{5},\ \tfrac{8}{5})$$

Ex. 4. *Find the equation of the circle which passes through the origin and belongs to the co-axal system whose limiting points are (1, 3) and (2, 6).*

Sol. The limiting points are the *point-circles* of the system. Their equations are

$$(x-1)^2+(y-3)^2=0$$

and $$(x-2)^2+(y-6)^2=0$$

∴ the equation of the co-axal system determined by the circles is

$$(x-1)^2+(y-3)^2+k\{(x-2)^2+(y-6)^2\}=0$$

$$(\because\ S+kS'=4)$$

This circle will pass through the origin if

$$10+40k=0 \qquad \text{or } k=-\tfrac{1}{4}.$$

Hence the required circle is

$$(x-1)^2+(y-3)^2-\tfrac{1}{4}[(x-2)^2+(y-6)^2=0$$

or $$3(x^2+y^2)-4x-12y=0$$

Exercise 16

1. Find the external common tangents to two circles

$$x^2+y^2=16$$

and $$x^2+y^2+6x-8y=0$$

2. Find the equations of the straight lines which touch both the circles

$$x^2+y^2=4 \text{ and } (x-4)^2+y^2=0.$$

Find also the co-ordinates of the centres of similitude.

3. Show that the common tangents to the circles

$$x^2+y^2+2x=0 \quad \text{and} \quad x^2+y^2-6x=0$$

form an equilateral triangle.

4. Find the common external tangents of the circles

$$x^2+y^2-2x-6y+9=0$$

and

$$x^2+y^2+6x-2y+1=0$$

Find the radical axis of the circles.

5. $x^2+y^2=4$ and $x^2+y^2-2x-3y=0$.
6. $x^2+y^2+4x+4y+7=0$ and $x^2+y^2-6x+2y-3=0$.
7. $x^2+y^2-bx+by+c=0$ and $ax^2+ay^2-a^2x+b^2y=0$.

8. Find the radical axis and the length of the common chord of the circles

$$x^2+y^2+ax+by+c=0$$

and

$$x^2+y^2+bx+ay+c=0.$$

(*P.U. 1957 S*)

9. Find the length of the common chord of the circles

$$x^2+y^2+2x+3y+1=0$$

and

$$x^2+y^2+4x+3y+2=0$$

10. Show that the circles

$$x^2+y^2-6x-6y+10=0 \quad \text{and} \quad x^2+y^2=2$$

touch and find the point of contact.

11. Prove that the circles

$$x^2+y^2-2x-4y+1=0$$

and

$$x^2+y^2+4x+4y-1=0$$

touch each other and find the equation of the common chord.

12. Show that the circles $x^2+y^2+4x+2y=15$ and $x^2+y^2=5$ touch *internally* and find their point of contact.

13. Prove that the two circles

$$x^2+y^2+2ax+c=0$$

and

$$x^2+y^2+2by+c=0$$

touch if $$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}.$$

Find the radical centre of the following circles :—

14. $x^2+y^2+11x+5x+7=0,\quad x^2+y^2+13x+6y+2=0,$
$$x^2+y^2+17x+3y-3=0.$$

15. $x^2+y^2+4x+7=0,\quad 2x^2+2y^2+3x+5y+9=0.$
$$x^2+y^2+y=0.$$

16. $x^2+y^2-4x+6y+2=0.\quad x^2+y^2-1=0,$
$$x^2+y^2-8x+12y+1=0.$$

17. $x^2+y^2+a_1x+b_1y+c=0,\quad x^2+y^2+a_2x+b_2y+c=0,$
$$x^2+y^2+a_3x+b_3y+c=0.$$

18. Find the limiting points of the co-axal system given by

(*i*) $x^2+y^2-6x-6y+4=0,\quad x^2+y^2-2x-4y+3=0.$ (*K.U. 1952*)

(*ii*) $x^2+y^2+2x+3y+1=0,\quad x^2+y^2+4x+3y+2=0.$ (*K.U. 1950*)

19. Find the equation of the circle which passes through the origin and belongs to the co-axal system whose limiting points are (1, 2) and (4, 3).

20. The point (2, 1) is a limiting point of the co-axal system of circles of which
$$x^2+y^2-6x-4y-3=0$$
is a member. Find the equation of the radical axis and the co ordinates of the other limiting point.

21. The origin is the limiting point of a co-axal system one of whose members is
$$x^2+y^2-2ax-2by+c=0\ ;$$
find the limiting point.

22. If the origin be at one of the limiting points of a co-axal system of which $x^2+y^2+2gx+2fy+c=0$ is a member,

show that the equation of the system of circles cutting them all orthogonally is

$$(x^2+y^2)(g+\mu f)+c(x+\mu y)=0$$

μ being a parameter. *(P.U. 1961)*

23. Prove that the polars of a given point *w.r.t.* a system of co-axals meet in another fixed point and the line joining these points is bisected by the radical axis of the system.

(P.U. 1948)

24. If A, B, C, be the centres of three co-axal circles and t_1, t_2, t_3 be the tangents to them from any point, prove that

$$BC.t_1{}^2+CA.t_2{}^2+AB.t_3{}^2=0.$$

25. If A, B, C be the centres and a, b, c the radii of three co-axal circles, prove that

$$a^2.BC+b^2.CA+c^2.AB+BC.CA.AB=0$$

(P. U. 1953)

J. & K. UNIVERSITY PAPERS

K.U. 1966

1. (*a*) If the point (x, y) is equidistant from the points $(a+b, b-a)$ and $(a-b, b+a)$, prove that $bx=ay$.

(*b*) Show that the area of the parallelogram whose sides are along the lines

$$y=mx+a,\ y=mx+b,\ y=nx+c$$

$$y=nx+d \text{ is } (a-b)(c-d)/(m-n)$$

(*c*) Find the value of λ so that the equation

$$x^2-\lambda ny+2y^2+3x-5y+2=0$$

may represent a pair of straight lines and find them.

2. (*a*) Find the equations of the circles which pass through the origin and cut off equal chords a from the lines $y=\pm x$.

(*b*) Obtain the equation of a system of co-axal circles in the simplest form.

3. (*a*) Show that all the chords of the curve

$$3x^2-y^2-2x+4y=0$$

which subtend a right angle at the origin pass through a fixed point.

(*b*) Show that the lines

$$y^2-2xy\sec\theta+x^2=0$$

make an angle θ with one another.

K.U. 1967

1. (*a*) Find the length of the perpendicular from a fixed point on a given straight line.

If p be the length of the perpendicular from the origin on the line

$$\frac{x}{a}+\frac{y}{b}=1$$

prove that
$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

(*b*) Find the equation to the line through the point of intersection of $5x-y+3=0$ and $3x+7y-6=0$ and the origin.

2. (*a*) Find the angle between the lines

$$ax^2+2hxy+by^2=0$$

(*b*) Show that if one of the straight lines given by the equation

$$ax^2+2hxy+by^2=0$$

coincides with one of those given by

$$a'x^2+2h'xy+b'y^2=0$$

then
$$(ab'-a'b)^2+4(ah'-a'h)(bh'-b'h)=0.$$

3. (*a*) Define co-axal system of circles. Find the equation to the system in the simplest form.

(*b*) Find the condition that the line $y=mx+c$ may be tangent to the circle $x^2+y^2=a^2$; show that the circle

$$x^2+y^2+4x-4y+4=0$$

touches the axes of co-ordinates.

ANSWERS

Exercise 1

1. 5. **2.** 13.

3. $\sqrt{a^2+b^2}$. **4.** $\sqrt{(a^2+2b^2+c^2-2ab-2bc}$.

5. $2a \sin \dfrac{\alpha-\beta}{2}$. **6.** $a(m_1-m_2)\sqrt{(m_1+m_2)^2+4}$.

7. $3 \pm 2\sqrt{15}$.

Exercise 2

1. $\frac{10}{7}, \frac{38}{7}$. **2.** $-2, -9$.

3. $(1, -\frac{4}{5})$; $(-11, 16)$. **4.** $(-5\frac{11}{12}, 2\frac{5}{12})$; $(-20\frac{1}{2}, 34\frac{1}{2})$.

5. $(-\frac{1}{3}, 0)$; $(-\frac{5}{3}, 2)$. **6.** $(-\frac{5}{2}, \frac{5}{2})$; $(1, 1)$; $(\frac{9}{2}, -\frac{5}{2})$.

7. 3, 8. **8.** $(0, 9)$; $(-12, -3)$; $(4, -7)$.

9. $(2, -1)$. **10.** $(-4, -8)$.

11. (*i*) $-5:2$ at $(-1, 0)$ (*ii*) $-4:1$ at $(0, 1)$.

12. (*i*) 3, 3, $\frac{3}{2}$. (*ii*) $\frac{1}{2}\sqrt{226}, \frac{1}{2}\sqrt{538}, \sqrt{202}$.

14. $(2, 4)$.

15. (*i*) $(-1, 0)$. (*ii*) $(14, 9)$.

Exercise 3

1. (*i*) $13\frac{1}{2}$.

(*ii*) $ab \sin \frac{1}{2}(\alpha-\beta) \sin \frac{1}{2}(\beta-\gamma) \sin \frac{1}{2}(\gamma-\alpha)$.

(*iii*) 42. (*iv*) 15.

(*v*) $a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$.

(*vi*) $\frac{1}{2}a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)$.

(*vii*) $\frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2) \div m_1m_2m_3$.

5. $k=-4$.

8. (*i*) 85. (*ii*) $\frac{11}{2}$.

Exercise 4

1. $y^2-4y-2x+5=0$.

2. (*i*) $2x+4y+3=0$. (*ii*) $x+y=7$. (*iii*) $x=y$.

3. $y=3x$. **4.** $15x^2-y^2+2ax=a^2$.

5. $x^2+y^2=3$.

6. (*i*) $2ax+k^2=0$.

(*ii*) $4x^2(c^2-4a^2)+4c^2y^2=c^2(c^2-4a^2)$.

(*iii*) $(n^2-1)(x^2+y^2+a^2)+2ax(n^2+1)=0$.

(*iv*) $(6a-2c)x=a^2-c^2$.

8. $x+3y-11=0$. **9.** $x^2+y^2=d^2$.

11. $(x^2+y^2)(x+y)=axy$. **12.** $a^2x^2+b^2y^2=a^2b^2$.

Exercise 5

1. $y=x+1$. **2.** $x-y-5=0$.

3. $x-\surd 3y2\surd -3=0$. **4.** $5y-3x+15=0$.

5. $2x+3y=6$. **6.** $6x-5y+30=0$.

7. (*i*) $x+y=11$. (*ii*) $y-x=1$.

8. $xy'+x'y=2x'y'$. **9.** $20y-9x=96$.

10. $\dfrac{x}{5\pm\sqrt{17}}+\dfrac{y}{3\mp\sqrt{17}}=1$.

11. $7y-6x=84$.

12. (*i*) 135°. (*ii*) 45°. (*iii*) 60°. (*iv*) 30°.

(*v*) The line is ∥ to the x-axis.

(*vi*) It is ⊥ to the x-axis.

13. $x+\surd 3y=4$. **14.** $y-x=5\surd 2$.

15. $y-x\surd 3=4$. **16.** $3x+4y=10$.

17. (*i*) $x \cos 30° + y \sin 30° = \frac{9}{2}$.

(*ii*) $x \cos 150° + y \sin 150° = \frac{7}{2}$.

(*iii*) $x \cos 210° + y \sin 210° = \frac{5}{2}$.

(*iv*) $x \cos 330° + y \sin 330° = \frac{11}{2}$.

18. (*i*) $y = \frac{2}{3}x + 2$. (*ii*) $y = -\frac{1}{2}x + \frac{3}{2}$.

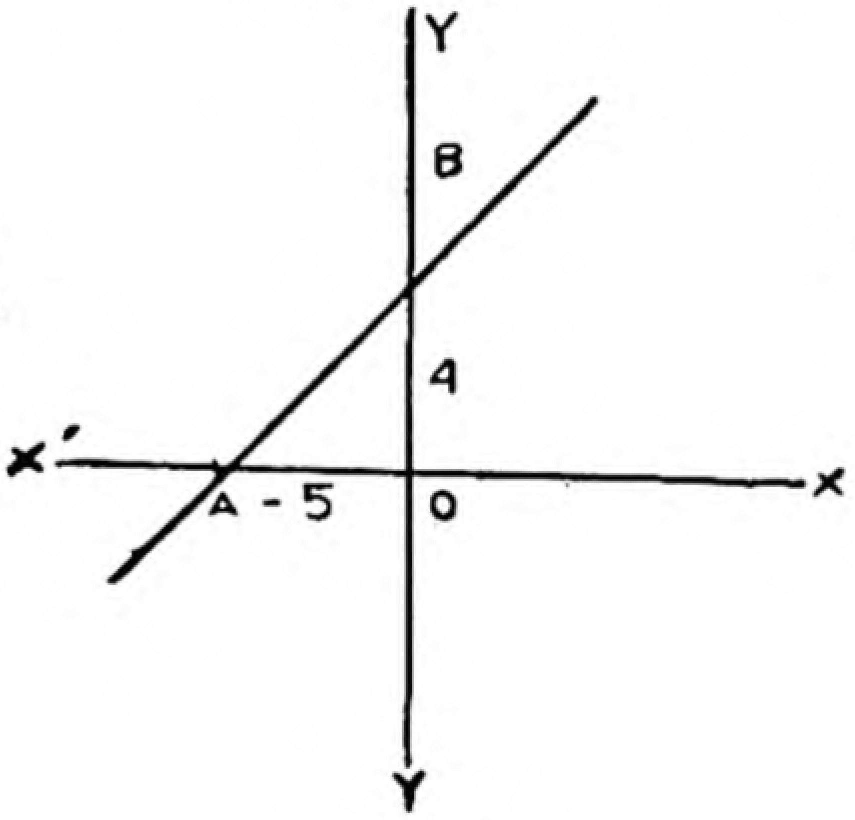
(*iii*) $y = -x \cos \alpha + p \operatorname{cosec} \alpha$.

19. (*i*) $\dfrac{x}{-\dfrac{5}{3}} + \dfrac{y}{\dfrac{5}{4}} = 1$. (*ii*) $\dfrac{x}{-5} + \dfrac{y}{4} = 1$.

(*iii*) $\dfrac{x}{2 \sin \alpha} + \dfrac{y}{2 \cos \alpha} = 1$.

(*iv*) $\dfrac{x}{3} + \dfrac{y}{9} = 1$. (*v*) $\dfrac{x}{8} + \dfrac{y}{-2} = 1$.

(i)



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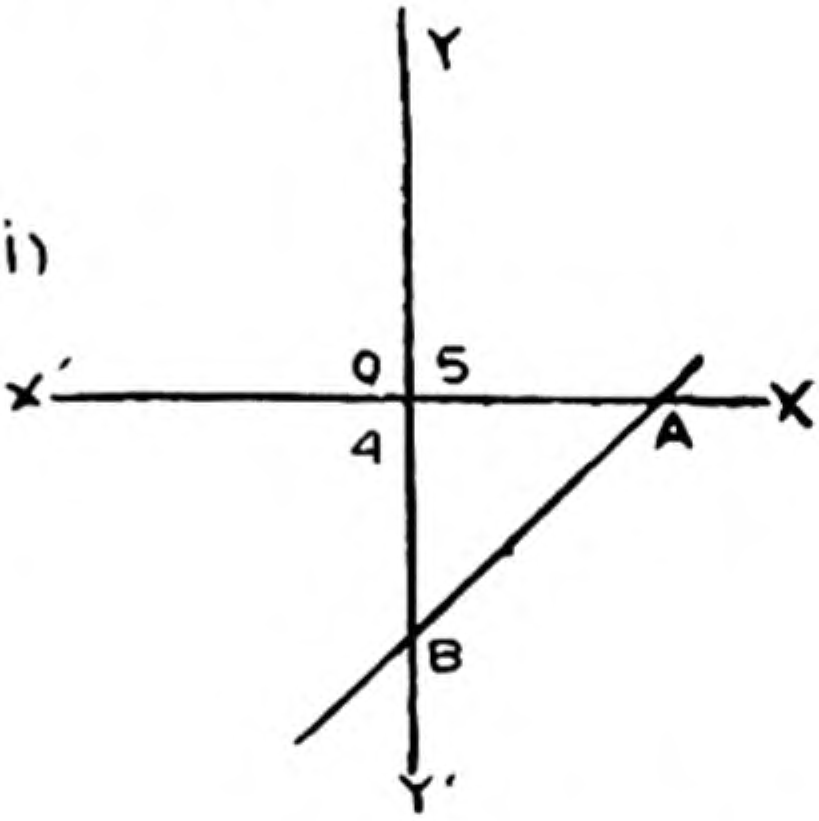

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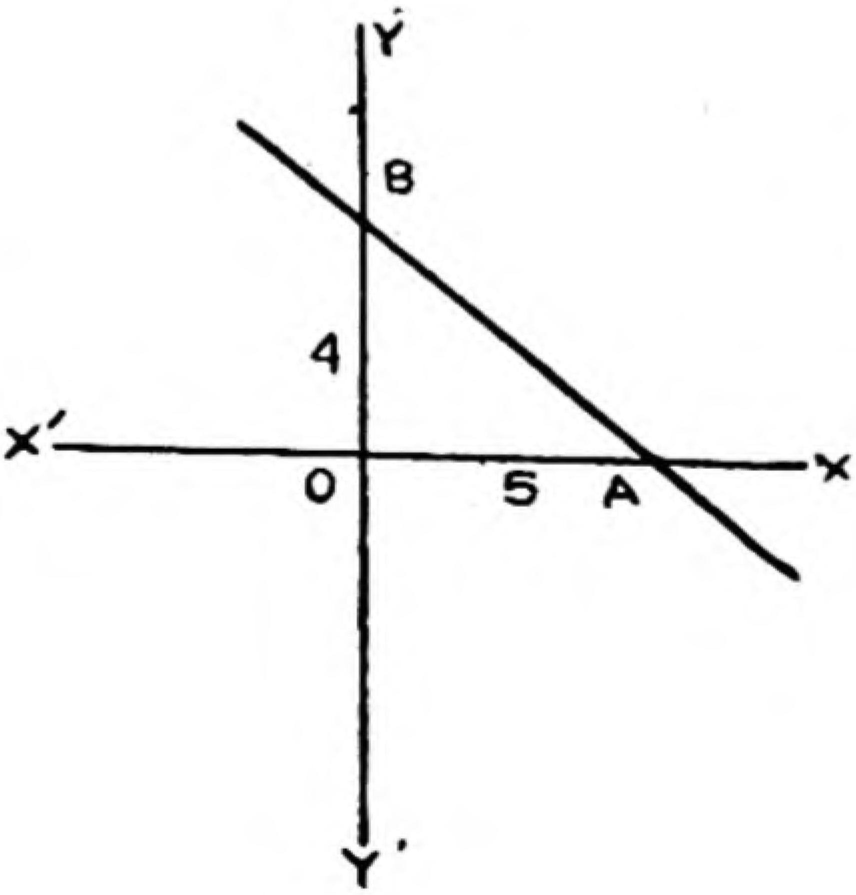

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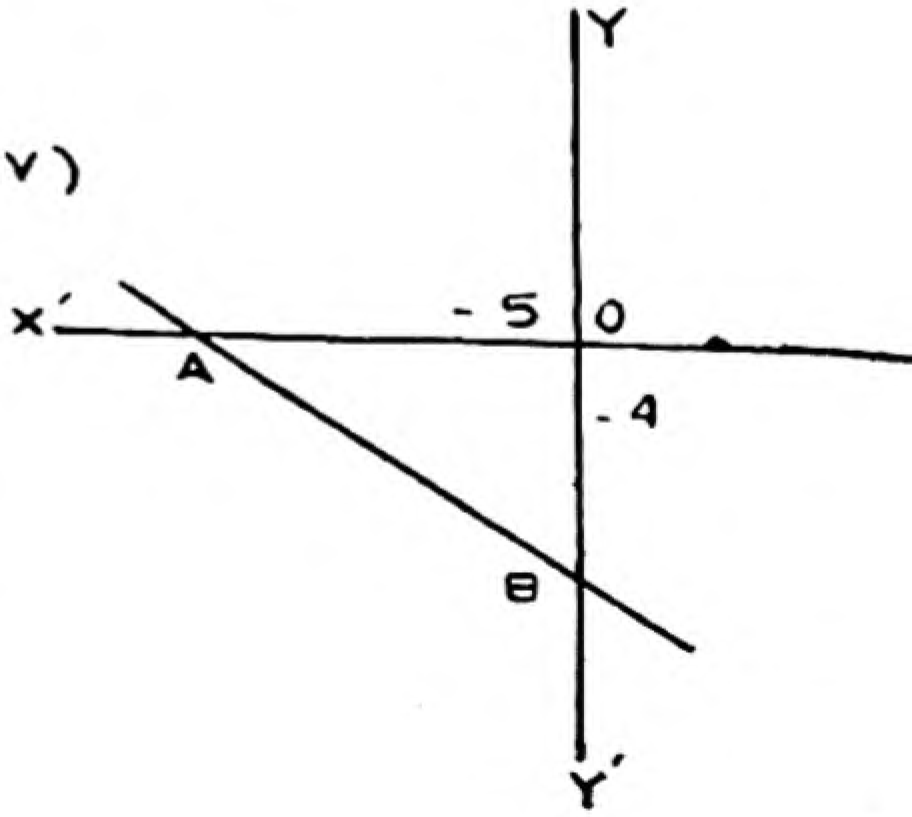

Exercise 6

1. $y=3x$. **2.** $y+2x=3$.

8. $y=x+2$, $y=-x$. **4.** $x=y$, $x+y=0$.

5. $x\cos\alpha+y\sin\alpha=p$.

6. (*i*) $x+y=1$. (*ii*) $x+y=0$.

(*iii*) $y-x=1$. (*iv*) $10x+7y=11$.

(*v*) $y(t_1+t_2)=2x+2at_1t_2$.

(*vi*) $t_1t_2y+x=a(t_1+t_2)$.

(*vii*) $x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=a\cos\frac{\beta-\alpha}{2}$.

(*viii*) $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

(*ix*) $bx\cos\frac{\alpha-\beta}{2}-a\sin y\frac{\alpha+\beta}{2}=ab\cos\frac{\alpha+\beta}{2}$.

(*x*) $(a-2b)x-by+b^2+2ab-a^2=0$.

9. (*i*) $x=2$, $x-2y+6=0$, $2x+2y+5=0$.

(*ii*) $5x-3y+2=0$, $x+12y-8=0$, $11x+6y-4=0$.

10. The line is bisected.

11. $(d-c)x+(a-b)y+be-ad=0$;
$(d-c)x+(b-a)y+ac-bd=0$.

13. The point is (1, 6). **14.** $\sqrt[4]{2}$.

15. $\frac{x-2}{\frac{\sqrt{3}}{2}}=\frac{y-3}{\frac{1}{2}}=r$ where $r=\frac{34}{2\sqrt{3}+3}$.

16. 15° or 75°.

Exercise 7

1. 90°. **2.** $\tan^{-1} 1\frac{23}{10}$.

3. $\tan^{-1}\frac{4}{3}$. **4.** $\tan^{-1}\frac{a^2-b^2}{2ab}$.

6. $2x+4y=11$. **7.** $3x-4y+6=0$.

8. $2x-3y+7=0$. **9.** $\mathrm{A}x+\mathrm{B}y=\mathrm{A}x'_1+\mathrm{B}y'$.

10. $3x+4y=18$. **11.** $7y-8x=118$.

12. $4y+11x=10$. **13.** $x+4y+16=0$.

14. $2x(a-a')+2y(b-b')=a^2-a'^2+b^2-b'^2$.

15. $8y=10x+29$.

16. $3x+y-5=0$, $4x-3y-5=0$, $2x+5y-5=0$.

17. $3x+y-17=0$, $x-3y+11=0$.

18. $y+2=0$, $\sqrt{3}x-y-(3\sqrt{3}+2)=0$.

Exercise 8

1. $\left(-\frac{11}{29}, \frac{41}{29}\right)$. **2.** $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

3. $\left[\frac{a}{m_1m_2}, a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right]$.

4. $\left[a \cos \frac{1}{2}(\varphi_1+\varphi_2) \sec \frac{1}{2}(\varphi_1-\varphi_2), a \sin \frac{1}{2}(\varphi_1+\varphi_2) \sec \frac{(\varphi_1-\varphi_3)}{2}\right]$.

5. The lines are parallel. **6.** $[at_1t_2, a(t_1+t_2)]$.

11. $k=5$.

12. (*i*) $(-4, -3)$. (*ii*) $(\frac{11}{13}, -\frac{7}{18})$.

13. $m_1(a_2-a_3)+m_2(a_3-a_1)+m_3(a_1-a_2)=0$.

14. $43x-29y=71$. **15.** $x-y=11$.

16. $y=3x$. **17.** $3x+4y=5a$.

18. $x+y+2=0$. **19.** $23x+23y=11$.

20. $13x-23y=64$.

21. (*i*) 1 sq. unit. (*ii*) $\frac{522}{180}$ sq. units.

25. $(-\frac{5}{2}, \frac{3}{2})$. **26.** $(\frac{3}{2}, \frac{5}{2})$.

Exercise 9

2. Same side. **3.** 2.

4. (*i*) 1. (*ii*) $\dfrac{a^2+ab-b^2}{\sqrt{a^2+b^2}}$. (*iii*) $\sqrt{h^2+k^2}$.

(*iv*) $5\sqrt{2}$. (*v*) $\dfrac{(a-b)^2}{\sqrt{a^2+b^2}}$.

5. 5 ; $(\frac{11}{17}, -\frac{7}{17})$. **8.** $\dfrac{5}{\sqrt{13}}$, $\dfrac{5}{\sqrt{13}}$, $\dfrac{5}{\sqrt{2}}$.

10. (*i*) $\frac{12}{5}$. (*ii*) $\dfrac{7}{\sqrt{5}}$.

(*iii*) $\dfrac{c-c'}{\sqrt{a^2+b^2}}\ (c>c')$. (*iv*) $\dfrac{c-d}{\sqrt{1+m^2}}\ (c>d)$.

11. $11x+3y-17=0$. **12.** $x-y-2=0$.

13. $8x-y-7=0$, $x+8y-9=0$.

14. $(\frac{7}{9}, \frac{14}{27})$; $\frac{14}{27}$.

Exercise 10

1. (*i*) 90°. (*ii*) $\tan^{-1}(\frac{3}{5})$. (*iii*) $\tan^{-1}(\frac{1}{13})$.

(*iv*) $\tan^{-1}\dfrac{\sqrt{\cos 2\theta}}{\sin\theta}$. (*v*) 45°.

2. (*i*) $8x^2-3xy-8y^2=0$. (*ii*) $x\ +6xy-y^2=0$.

(*tii*) $x^2-y^2=0$. (*iv*) $x^2-y^2=0$.

8. $bx^2-2hxy+ay^2=0$.

Exercise 11

1. $(3, 2)$; $\tan^{-1}\frac{3}{5}$. **2.** $(\frac{2}{5}, -\frac{4}{5})$; 45°.

3. $(2, 1)$; $\tan^{-1}\frac{7}{17}$. **4.** $(3, 2)$; 90°.

5. $\dfrac{6}{\sqrt{10}}$.

6. (*i*) $-\frac{5}{2}$ or $-\frac{10}{3}$. (*ii*) 1. (*iii*) -3.

PREFACE

This book has been written with a view to providing a suitable text-book on Co-ordinate Geometry for the students of the first year of three-year degree course. It has been written in strict accordance with the syllabus of the J. & K. University, and no pains have been spared to present the subject-matter in to understand form. Accordingly, various Articles have plained by means of a number of solved examples, most of which have been taken from University Papers. It is hoped that even the average student will easily dispense with "guides" on the subject if he tries to grasp the subject-matter as given in the book.

Suggestions for improvement will be thankfully acknowleged.

Central Jail Road,
Srinagar-3
March, 1967.

G. Mohi-ud Din.

J. & K. UNIVERSITY SYLLABUS

IN

CO-ORDINATE GEOMETRY

For the first year of 3-year Degree Course
for 1967 and after.

Rectangular Co-ordinates, Point, Straight line inc. joint equation of two st. lines, Circle including Co-axal circles.

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Exercise 12

1. $\sqrt{2}$; $(4, 8)$. **2.** $\frac{\sqrt{101}}{14}$; $\left(\frac{2}{7}, \frac{1}{14}\right)$.

3. $\frac{\sqrt{b^2+c^2}}{2a}$, $\left(\frac{b}{2a}, \frac{c}{2a}\right)$. **4.** $\sqrt{61}$; $(2, 4)$.

5. $(\frac{5}{6}, 1)$; $\frac{1}{6}\sqrt{13}$. **6.** $\frac{\sqrt{5}}{2}k$; $\left(\frac{k}{2}, 0\right)$.

7. $\sqrt{f^2+g^2}$; $(g, -f)$. **8.** $x^2+y^2+2x-4y-4=0$.

9. $x^2+y^2+10x+12y=39$.

10. $x^2+y^2+2ax+2by+2b^2=0$.

11. $x^2+y^2-2ax+2by=2ab$.

12. $x^2+y^2-ax-by=0$. **13.** $x^2+y^2-22x-4y+25=0$.

14. $x^2+y^2-5x-y+4=0$.

15. $b(x^2+y^2)-(a^2+b^2)x+(a-b)(a^2+b^2)=0$.

17. $x^2+y^2+5y-30=0$. **18.** $15(x^2+y^2)-94x+18y+55=0$.

19. $x^2+y^2-6x-8y+15=0$.

20. $x^2+y^2-3x-4y=0$. **21.** $x^2+y^2-hx-ky=0$.

22. $x^2+y^2-10x-10y+25=0$.

23. $x^2+y^2\pm 2ax\pm 2ay+a^2=0$.

24. $x^3+y^2-x-2y+1=0$.

25. $(x-a)(x-c)+(y-b)(y-d)=0$.

26. $x^2+y^2-3x-y-22=0$.

27. $x^2+y^2-6x+4=0$. **28.** $x^2+y^2-5x+3y-22=0$.

Exercise 13

1. $6x+8y+25=0$. **2.** $x-4y+9=0$.

3. $x+3y-10=0$; $x-3y-10=0$.

4. $gx+fy=0 \because c=0$. **5.** $x+y+1=0$.

6. $12x+5y=0$. **7.** $x-y-3=0$.

8. $4x-3y=0$. **9.** $4x+y+2=0$.

10. $ax-by=0$.　　**11.** $(0, 5), (-4, 3)$.

12. $(0, a), \left[-\frac{2am}{1+m^2}, \frac{a(1-m^2)}{1+m^2}\right]$.

13. The line touches the circle at $(-1, -1)$.

14. $(-\frac{12}{5}, -\frac{16}{5})$.　　**15.** $(7, 3) ; (2, 8)$.

16. $a^2(A^2+B^2)=C^2 ; \left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right)$.

17. $Aa+Bb+C=\pm c\sqrt{A^2+B^2}$.

18. $2\sqrt{r^2-\frac{a^2b^2}{a^2+b^2}}$.

19. $2\sqrt{a^2-\frac{q^2}{1+p^2}} ; \left(-\frac{pq}{1+p^2}, \frac{q}{1+p^2}\right)$.

20. $\left(\frac{1}{5}, \frac{7}{5}\right), (-1, -1) ; \frac{6}{\sqrt{5}}$.

21. $\frac{5}{4}, -\frac{45}{4}$.　　**22.** $x^2+y^2-2x+4y-3=0$.

23. $\sqrt{3}x-y+4=0, \sqrt{3}x-y-4=0$.

24. (*i*) $x+2y+2\sqrt{5}=0, x+2y-2\sqrt{5}=0$.

(*ii*) $x+2=0, x-2=0$.

25. $3x-4y+20=0, 3x-4y-10=0$.

26. (*i*) $x+y+3\sqrt{2}=0, x+y-3\sqrt{2}=0$.

(*ii*) $4x+3y+5=0, 4x+3y-25=0$.

Exercise 14

1. The point lies outside the circle.
2. The point lies outside the circle.
3. The point lies outside the circle.
4. The point lies on the circle.
5. The point lies on the circle.
6. The point lies inside the circle.
7. $x-3y-10=0 ; 3x+y-10=0$.

8. $y-2=0,\ 4x-3y-10=0$; $(0, 2),\ (\frac{8}{5}, -\frac{6}{5})$.

9. $2x-y+1=0,\ x+2y-2=0$.

10. $\frac{1}{4}\sqrt{46}$. **11.** 9.

12. $\sqrt{2a^2+2ab+b^2}$.

13. $28x^2+33xy-28y^2-715x-195y+4225=0$.

14. $125x^2-64xy+3y^2-664x+226y+763=0$.

15. $x-2y+5=0$. **16.** $2x-3y+13=0$.

18. $x^2+y^2-hx-ky=0$. **20.** $5x+9y+30=3$.

23. $x+2y=7$. **24.** $8x-2y=11$.

25. $x=0$. **26.** $by-ax=a^2$.

27. $(5, 10)$. **28.** $(\frac{3}{5}, -\frac{3}{10})$.

29. $(1, -2)$. **30.** $x'^2+y'^2=2a^2$.

Exercise 15

1. $\cos^{-1}\frac{5}{6}$.

2. $\cos^{-1}\dfrac{h^2+k^2-[(c-a)^2+(d-b)^2]}{2hk}$.

5. $2(aa'+bb')=c+\lambda c'$. **8.** $x^2+y^2-4x-4y=0$.

9. $7x^2+7y^2-8x-8y-12=0$.

11. $x^2+y^2-1=0$ and $x^2+y^2-\frac{15}{4}x-1=0$.

12. $x^2+y^2-6x-4y-44=0$.

13. $\dfrac{(a+b)^2-8c}{2\sqrt{2}}$. **14.** $\sqrt{6}$.

15. $2(x^2+y^2)+2x+6y+1=0$,

16. $2ab\div\sqrt{a^2+b^2}$.

17. (*i*) $2x^2+2y^2-10x-5y=0$.

(*ii*) $x^2+y^2+4x-7y+5=0$.

(*iii*) $x^2+y^2-3ax+by=0$.

Exercise 16

1. $(6-\sqrt{6})x+4y-4(2-3\sqrt{6})=0$; $(6+\sqrt{6})x+4y-4(2+3\sqrt{6})=0$.

2. $3x \pm \sqrt{7}\, y - 8 = 0$, $x \pm \sqrt{15}\, y - y\, 8 = 0$; $(8, 0)$, $(\frac{8}{8}, 0)$.

4. $x - \sqrt{15}\, y - 8 = 0$, $x + \sqrt{15}\, y - 8 = 0$.

5. $2x - 3y - 4 = 0$.

6. $5x - 3y + 5 = 0$.

7. $ax + by + \dfrac{ac}{a-b} = 0$.

8. $x - y = 0$; $\sqrt{\dfrac{(a+b)^2 - 8c}{2}}$.

9. $2\sqrt{2}$.

10. $(1, 1)$.

11. $3x + 4y - 1 = 0$.

12. $(2, 1)$.

14. 2, 1.

15. -2, -1,

16. The radical axes do not meet.

17. 0, 0.

18. (*i*) $(-1, 1)$, $(\frac{1}{5}, \frac{8}{5})$. (*ii*) imaginary.

19. $x^2 + y^2 - x - 3y = 0$.

20. $x + y + 4 = 0$; $(-5, -6)$.

21. $\left(\dfrac{ca}{a^2+b^2}, \dfrac{cb}{a^2+b^2}\right)$.

The End

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